Astrophysics Introductory Course

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Chapter 3

Theory of Stellar Structure

3.1 The basic equations of stellar structure: I

The goal is understand the inner structure of stars, their equilibrium configurations, nuclear burning, energy transport and evolution. In the standard model, we assume that a star can be treated as a **spherically symmetric gas sphere (no non-radial motions, no magnetic fields)**. This is a reasonably good approximation for most stars. The task is solved within the approximation when we have determined:

$$M(r),\ \rho(r),\ P(r),\ T(r),\ L(r)$$

Which equations constrain these parameters?

- (1) Hydrostatic equilibrium:

$$\frac{dP(\rho,T)}{dr} = -G\rho(r)\frac{M(r)}{r^2}$$

- (2) Mass-density relation:

$$dM(r) = 4\pi r^2 \rho(r) dr$$

- (3) Radial luminosity profile, i.e. the luminosity produced by nuclear burning in a shell of radius r and thickness dr

$$dL(r) = 4\pi r^2 \varepsilon_v(\rho, T) dr$$
$$= 4\pi r^2 \varepsilon(\rho, T) \rho(r) dr$$

$\varepsilon_v(\rho, T)$ denotes the energy production rate per volume, whereas $\varepsilon(\rho,T)$ is the energy production rate per mass. They can be determined by considering all nuclear reaction rates at a given temperature and density.

- (4) Equation of energy transport (important! e.g.: inhibiting the energy transport would imply zero luminosity and the explosion of the star).

We have 5 unknown parameters and 4 differential equations. However, one further constraint is the **equation of state**, which links ρ, P and T. For normal stars we have:

- **Equation of state (normal stars):**

$$P(\rho, T) = \frac{k_B}{\mu m_p} \rho T$$

So, we finally have 5 equations for 5 unknowns and we should be able to solve the problem in a unique way. The uniqueness of the solution is claimed in the:

Russel-Vogt-Theorem: *For a star of given chemical composition and mass there exists only one equilibrium configuration which solves the boundary problem of stellar structure.*

In this generality, the theorem is not proven. Local uniqueness can be shown, however. Also, the theorem is based on the wrong assumption that the chemical composition of a star is homogenous.

The two equations which require further elaboration are equations (3) and (4).

3.2 Nuclear energy production

3.2.1 Gravitational and chemical energy production

- Age of the Sun (from Age of the Earth and Meteorites):

$$\tau_\odot \sim 4.6\times10^9\, yr$$

- Luminosity must have been largely constant over this time; today 1g solar material produce 2×10^{-7}W; over 5×10^9yrs this implies 3×10^{10}Ws per g.

- Chemical processes can at most produce 10^4Ws per g and would limit the sun's age to 10^4yrs.

- Gravitational Energy (Helmholtz 1854, Kelvin 1861) does not suffice either: Starting from E = 0 at infinity we obtain via contraction and virialisation to a radius r:

$$E = E_{pot} + E_{therm} = -E_{therm} = -\frac{1}{2}\,|\,E_{pot}\,| \sim \frac{GM_\odot^2}{2R_\odot}$$

yielding the **Kelvin-Helmholtz time-scale**:

$$\tau_{grav} \approx \frac{E_{pot}}{L_\odot} \approx 10^7\, yr$$

3.2.2 Basics of nuclear reactions

Already in 1920, Eddington realizes that **nuclear fusion** may be the solution to the energy production of stars* (the processes are later elaborated in detail by Bethe, von Weizsäcker and others):

$$4p \mapsto {}^4He$$

The mass difference between 4 protons and the He-nucleus is:

$$\Delta m = 4m_H - m_{He}$$

corresponding to an energy of:

$$\Delta E = \Delta mc^2 = 26.72 MeV = 4.288 \times 10^{-5}\, erg$$

* Eddington writes in 1920: If, indeed, the sub-atomic energy in the stars is being freely used to maintain their great furnaces, it seems to bring a little nearer to fulfillment our dream of controlling this latent power for the well-being of the human race---or for its suicide.

Consequently, the Sun's resource of nuclear energy is: $E_{nuc} = \frac{\Delta mc^2}{4} N_p$

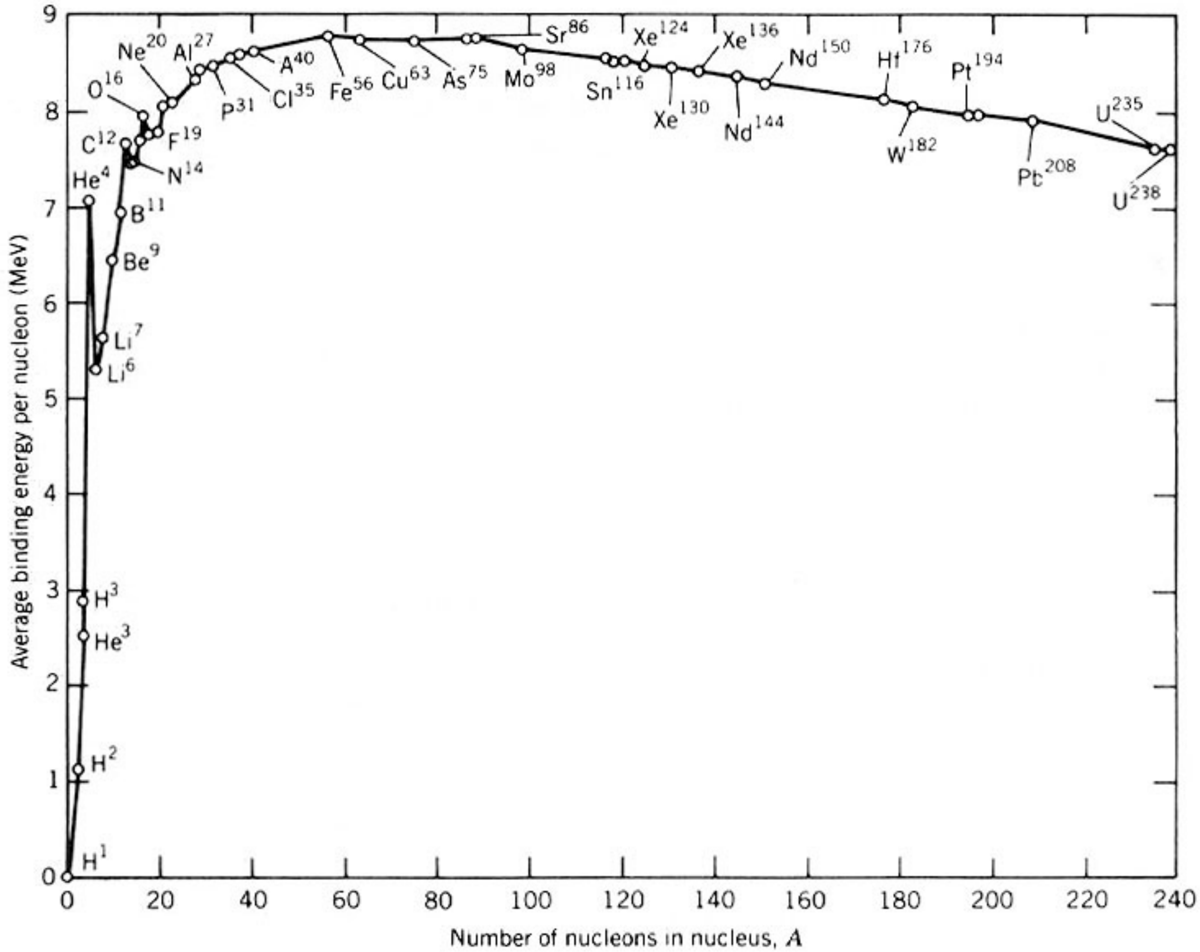
The number of protons in the Sun is: $N_p = \frac{3}{4} M_\odot / m_p$ and yields the nuclear time scale:

$$\tau_{nuc} = \frac{E_{nuc}}{L_\odot} \approx 10^{11}\, yr$$

As the fusion temperature is reached only in the core and mixing is not happening, only 10% of the suns mass is available for fusion. This results in a lifetime for the sun of 10^{10}years.

Stars similar to Sun live for:

$$\tau_* = 10^{10} \frac{M_* / M_\odot}{L_* / L_\odot} yr$$



Binding energy per nucleon: The energy release by nuclear fusion is highest for hydrogen burning and decreases for heavier elements. Beyond iron, fusion cannot produce energy anymore but needs energy input.

3.2.3 Nuclear reaction probabilities

- The sun is in hydrostatic equilibrium:

$$\frac{dp}{dr} = -\rho(r)\frac{GM(r)}{r^2}$$

which can be crudely approximated by:

$$\frac{p}{R_\odot} \sim \rho \frac{GM_\odot}{R_\odot^2}$$

Inserting the Ideal Gas Law:

$$p = kT\frac{\rho}{\mu m_p}$$

results in:

$$T_{c,\odot} \sim \frac{\mu m_p}{k}\frac{GM_\odot}{R_\odot} \approx 2\times 10^7\,K$$

The real value actually is 1.3×10^7. This implies a kinetic energy of the protons of :

$$E_{kin,p} = \frac{3}{2}kT_{c,\odot} \approx 1.7\,keV$$

This is much smaller than the **Coulomb Barrier** between the protons which corresponds to about 1 MeV. The classical fusion probability therefore is:

$$\mathrm{Prob}_{class} \sim e^{-1000} \approx 10^{-434}$$

which corresponds to the fraction of protons in the extreme Maxwell tail. As we have only 10^{57} Nucleons in the Sun, classically no fusion will take place.

- The solution of this riddle was found by **Gamow**: quantum mechanical tunneling. The tunneling probability for two nucleons is:

$$\mathrm{Prob}_{QM} = e^{-b/\sqrt{E}}$$

with

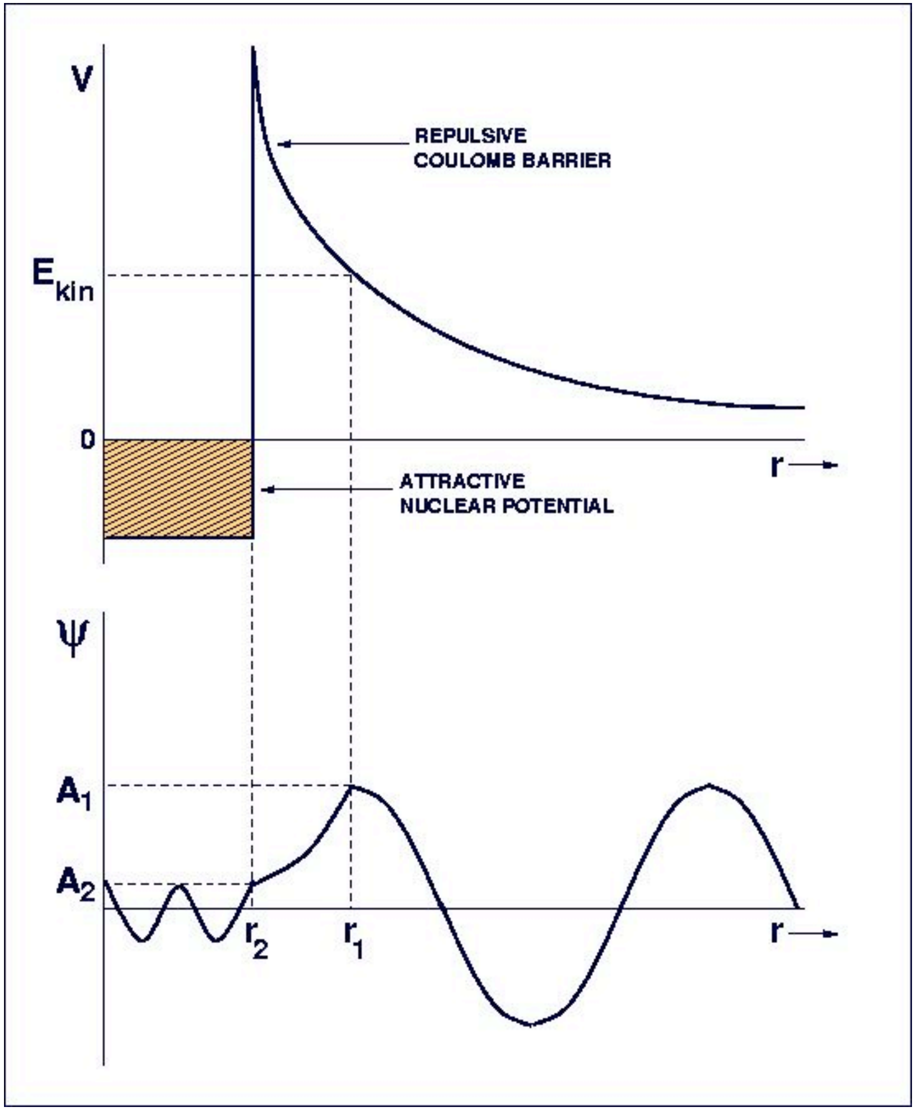
$$b = 31.28 Z_1 Z_2 \left(\frac{A_1 A_2}{A_1 + A_2} \right)^{1/2} (keV)^{1/2}$$

- For the p-p reaction in the Sun we have: $T = 10^7$, $E = kT$, $Z_1 = Z_2 = 1$, $A_1 = A_2 = 1$ which gives:

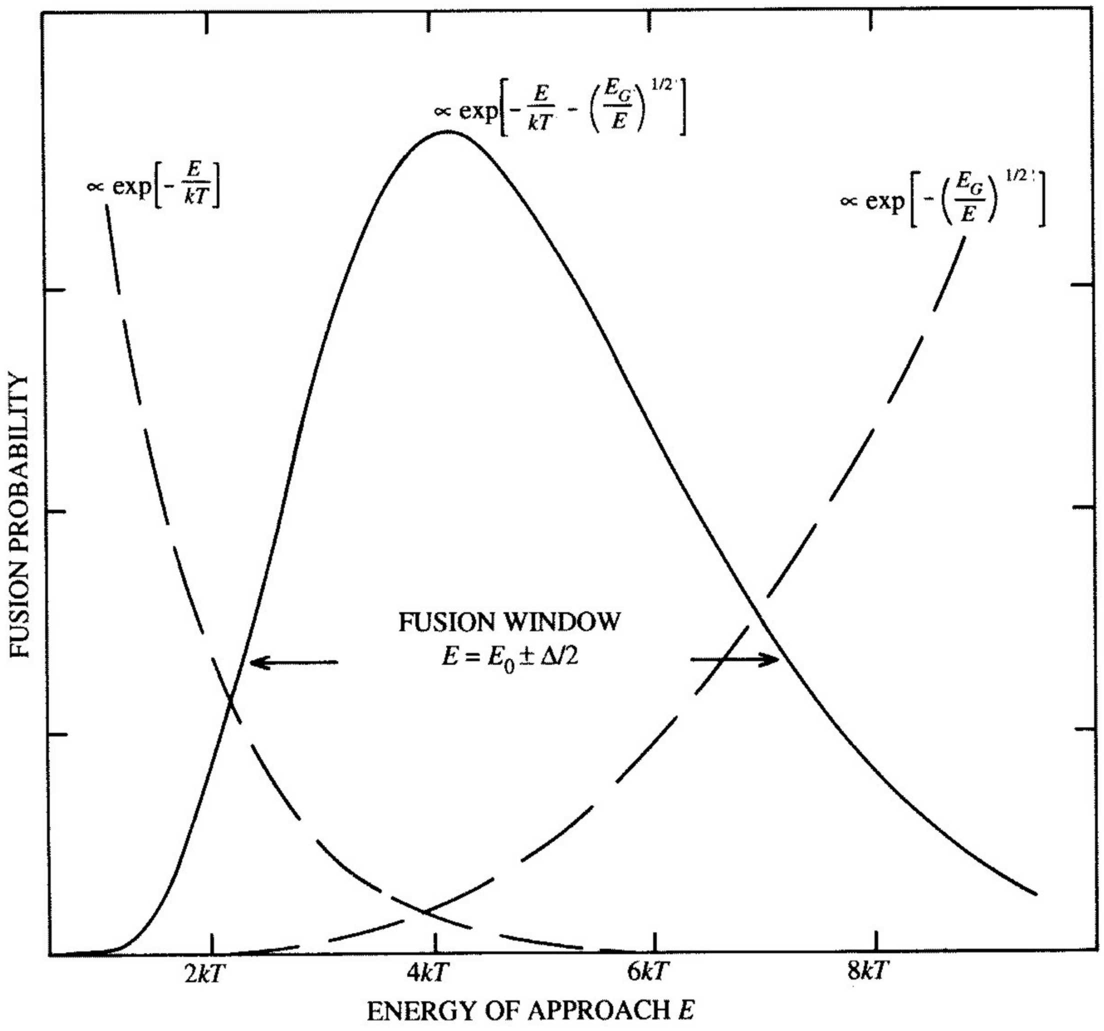
$$\mathrm{Prob}_{QM} \sim 10^{-10}$$

- So, this works! The formula also shows that it is easier to merge light nucleons and that heavier nucleons require higher T

The Coulomb Barrier



Energy Window for Nuclear Reactions: Gamow Peak



3.2.4 Hydrogen burning

Stars contain about 90% hydrogen, 10% helium and 1% heavy elements (by number and at birth). 90% of all stars burn hydrogen.

The two ways to burn hydrogen are:

- pp–chain

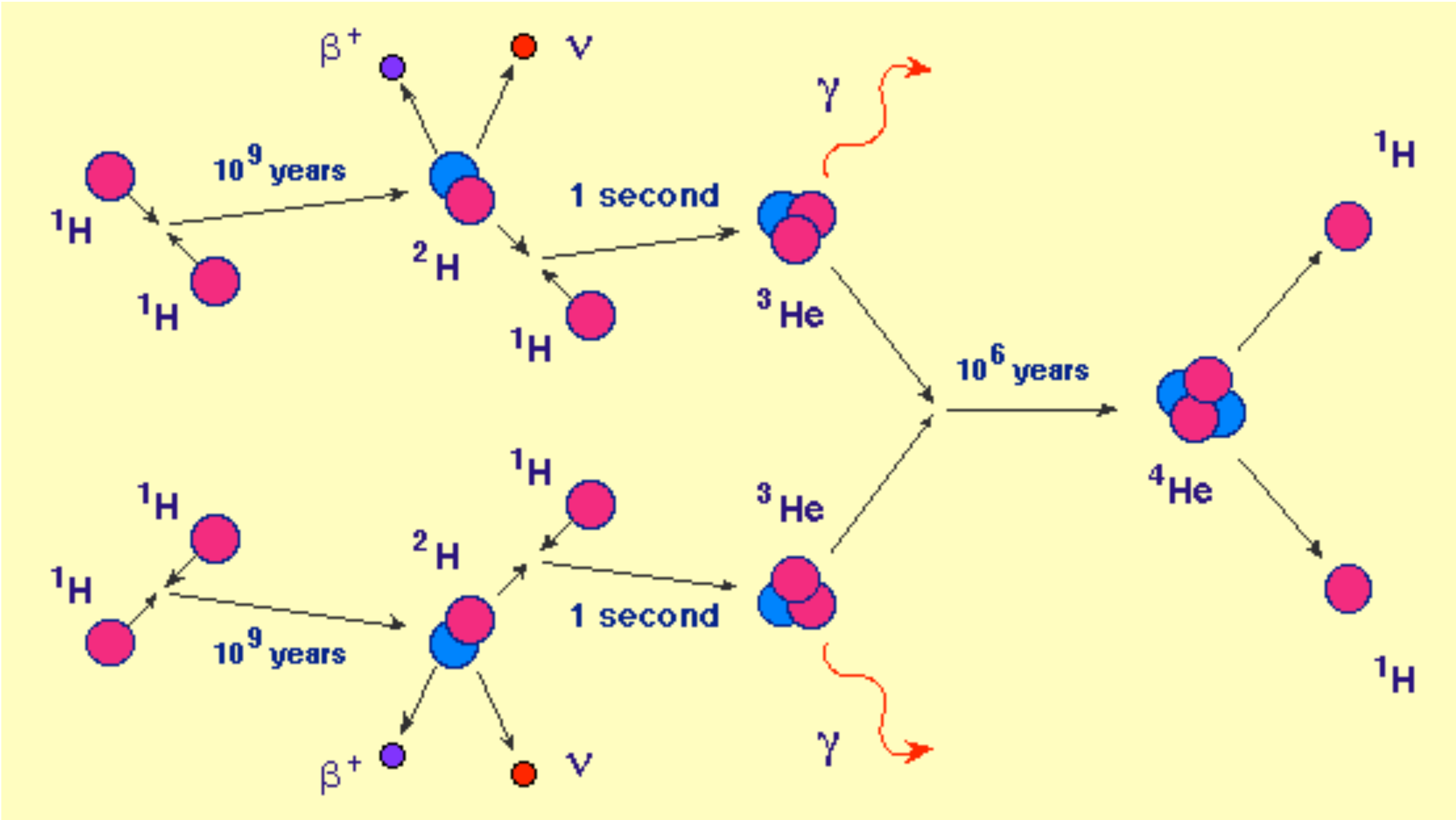
$$^{1}H(p,e^{+}\nu_{e})^{2}D(p,\gamma)^{3}He(^{3}He,2p)^{4}He+26.21MeV$$

- CNO–cycle

$$^{12}C(p,\gamma)^{13}N(e^{+}\nu)^{13}C(p,\gamma)^{14}N(p,\gamma)^{15}O(e^{+}\nu_{e})^{15}N(p,\alpha)^{12}C+25.0MeV$$

The difference in the released energy is due to neutrinos which escape from the star's core without interaction.

Rates of the proton-proton chain



The average time required for a nucleus to undergo each step of this sequence in a typical stellar interior is indicated in the figure shown above. Thus, for example, a hydrogen nucleus waits on the average 1 billion years before it undergoes an interaction with another hydrogen nucleus to initiate the sequence! Since all other steps require much less time than this, it is this initial step that controls the rate of the reaction.

This incredibly small rate nevertheless accounts for the luminosities of normal stars because there are so many hydrogen atoms in the core of a star that at any one instant many are undergoing the reactions of the pp–chain.

The pp–chain is the slowest reaction (10^9 years for the sun), this means that the sun can burn hydrogen for about 10^9 years before conditions change significantly.

There exist two further variants of the pp-chain going through Beryllium and Boron. All three variants and the CNO cycle (see below) produce **neutrinos** of different energies which can nowadays be used to test the accuracy of our model of the solar reactor. Because neutrinos only participate in weak interactions ($\sigma_v \sim 10^{-43}$ cm^2), the mean free path l_v of the neutrinos in the sun is: $l_v = (n\sigma_v)^{-1} = 10^{17}$ cm (n = particle density in the solar centre ~ 10^{26} cm^{-3}). Therefore, neutrinos offer the possibility to observe the solar reactor directly.

3.2.5 The p-p-chain and the neutrinos from the sun

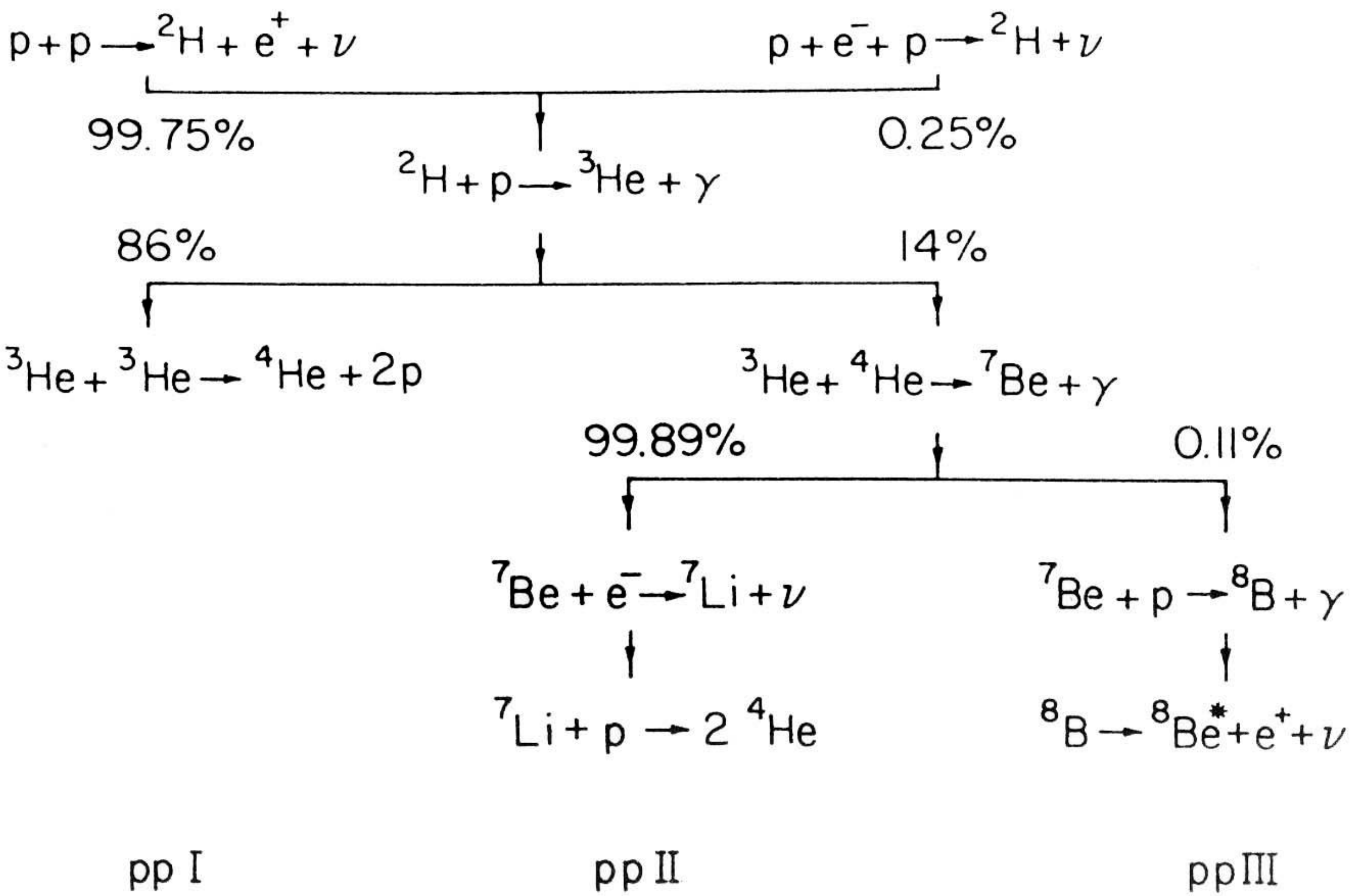


Table 1 Neutrino fluxes predicted by the Bahcall/Pinsonneault (with and without He diffusion) and Turck-Chièze/Lopez standard solar models

Source	E_ν^{max} (MeV)	Flux (cm^{-2} s^{-1}) BP (with diffusion)	BP (without)	TCL
$p + p \rightarrow {}^2H + e^+ + \nu$	0.42	6.00E10	6.04E10	6.03E10
${}^{13}N \rightarrow {}^{13}C + e^+ + \nu$	1.20	4.92E8	4.35E8	3.83E8
${}^{15}O \rightarrow {}^{15}N + e^+ + \nu$	1.73	4.26E8	3.72E8	3.18E8
${}^{17}F \rightarrow {}^{17}O + e^+ + \nu$	1.74	5.39E6	4.67E6	
${}^8B \rightarrow {}^8Be + e^+ + \nu$	~15	5.69E6	5.06E6	4.43E6
${}^3He + p \rightarrow {}^4He + e^+ + \nu$	18.77	1.23E3	1.25E3	
${}^7Be + e^- \rightarrow {}^7Li + \nu$	0.86(90%) 0.38(10%)	4.89E9	4.61E9	4.34E9
$p + e^- + p \rightarrow {}^2H + \nu$	1.44	1.43E8	1.43E8	1.39E8

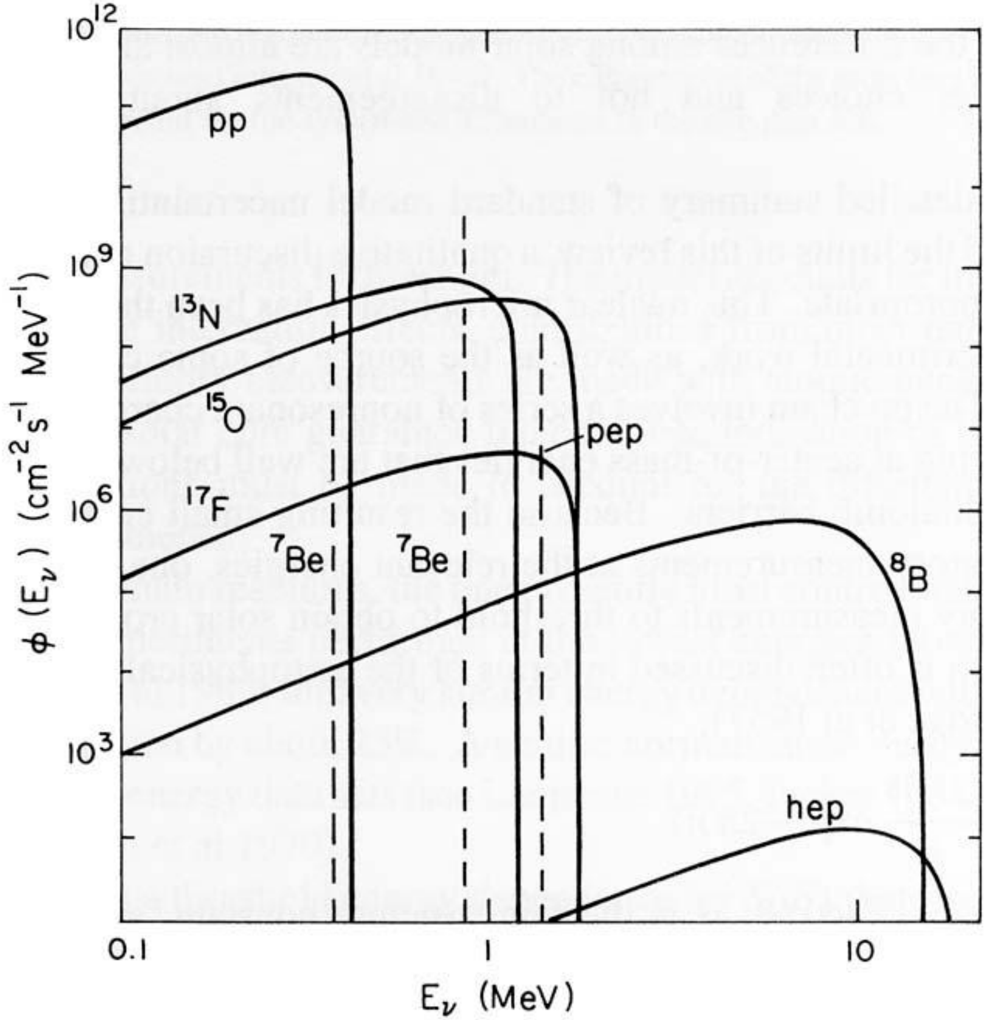
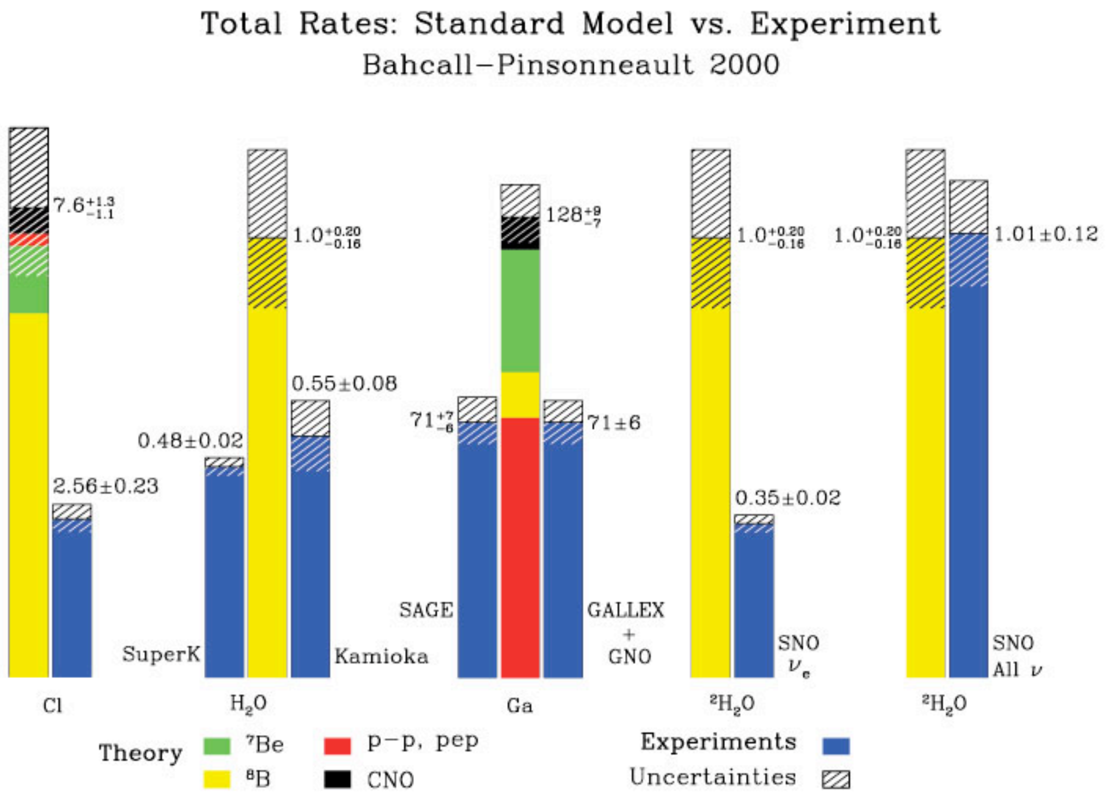
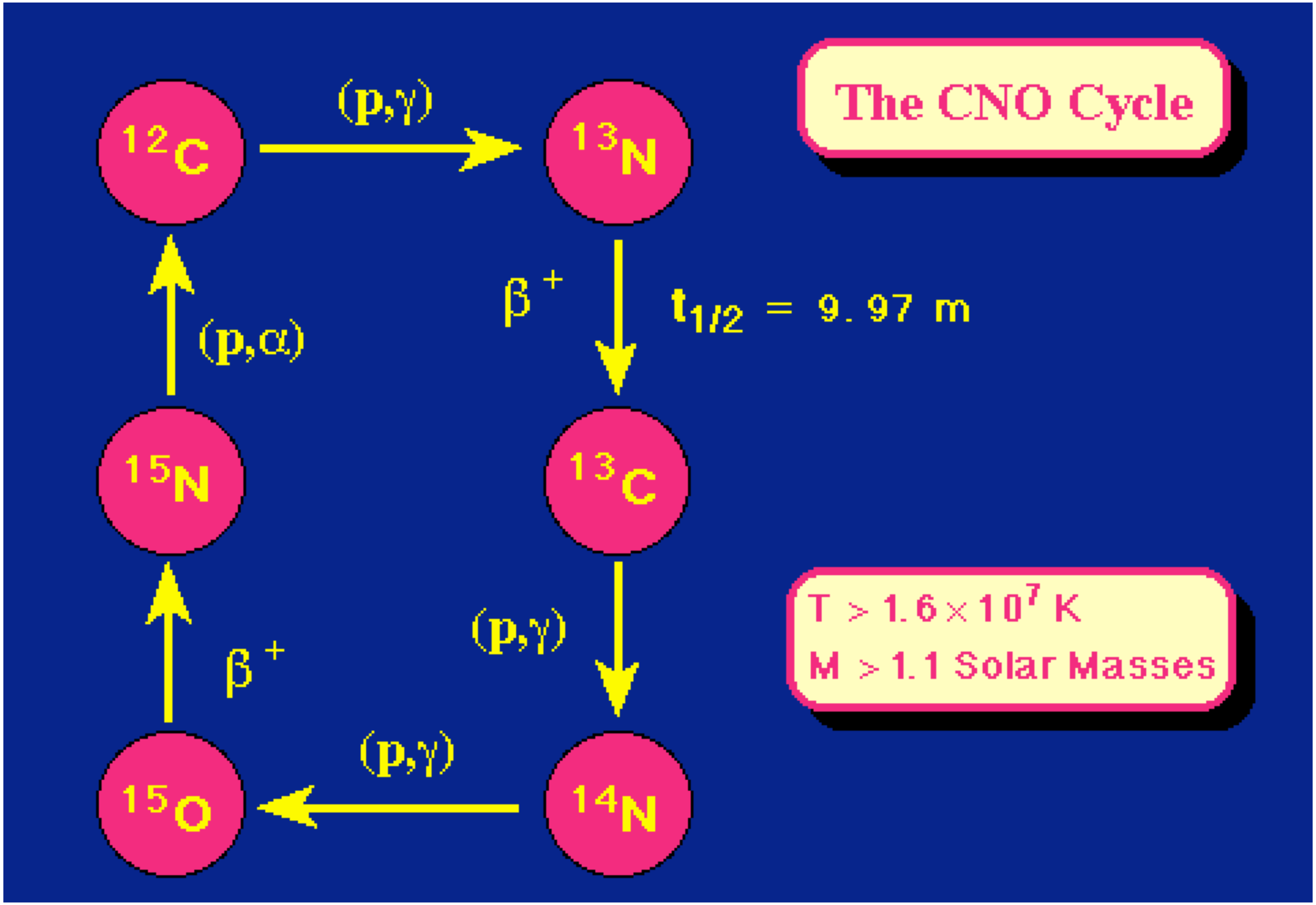


Figure 2 The flux densities (*solid lines*) of the principal β decay sources of solar neutrinos of the standard solar model. The total fluxes are those of the BP SSM. The ^{7}Be and pep electron capture neutrino fluxes (*dashed lines*) are discrete and given in units of cm^{-2} s^{-1}.



The flux of electron neutrinos from the sun is less than half of what is expected from the solar model. On the other hand, **helioseismology** has shown the validity of the solar model. The only way out are neutrino oscillations (e.g., electron neutrinos change to muon or tau neutrinos during trip to the earth because they have a small mass). The recent SNO experiment is sensitive to all neutrino flavors and has indeed shown that the total neutrino flux is very close to the theoretical prediction.

3.2.6 CNO Cycle

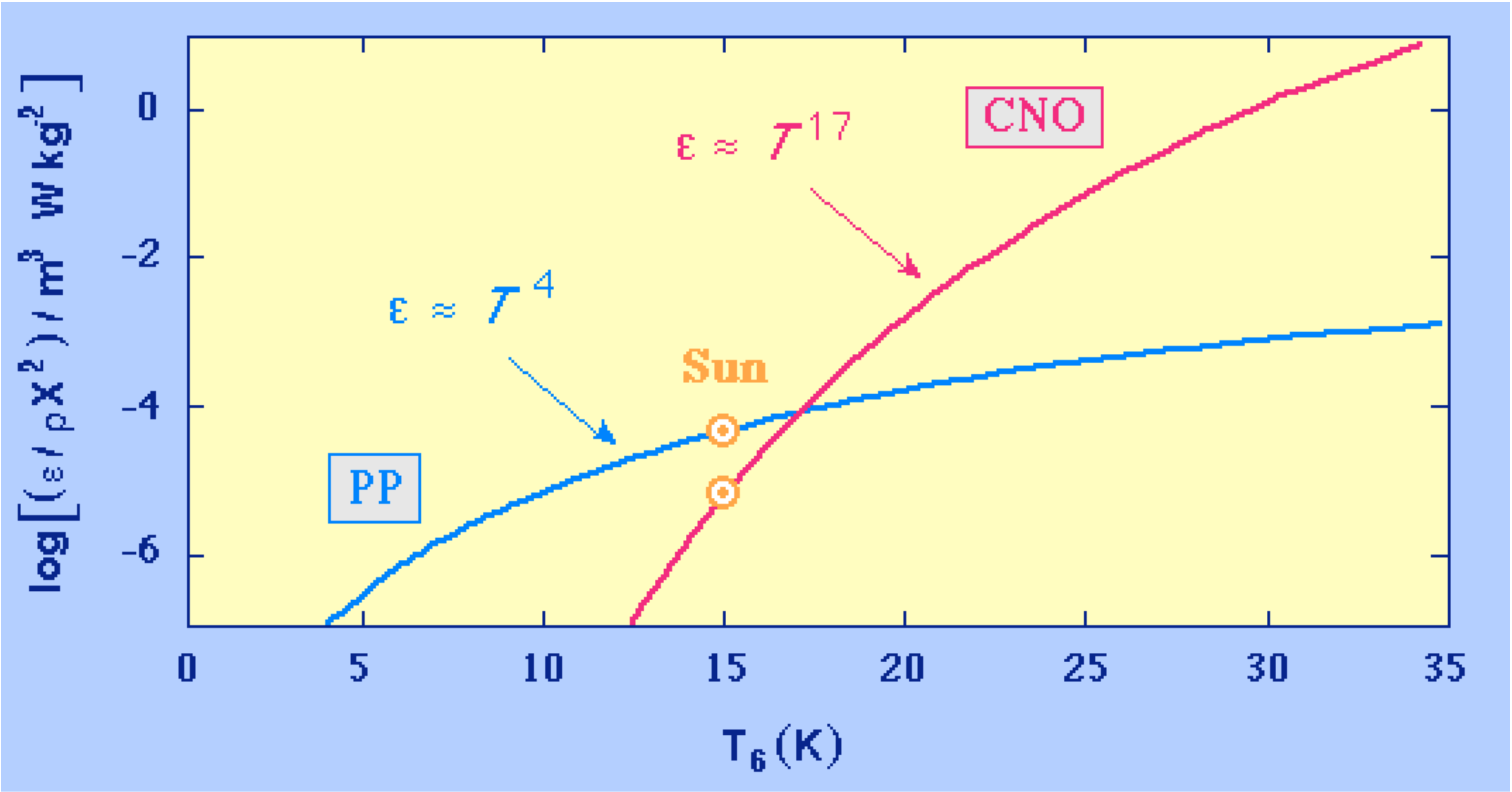


The net effect of the CNO cycle is to convert hydrogen to helium (the alpha particle emitted in the last step). Carbon, nitrogen and oxygen act as catalysts, i.e. they are needed to drive the reaction but are not burned or produced themselves.

The time scale for the CNO cycle at $T = 10^7$ K and $\rho = 10^2$ g/cm^2 is around 10^7 years.

In the case of the Sun, detailed modelling shows that it is producing about 98-99% of its energy from the pp–chain and only about 1% from the CNO cycle. However, if the Sun were only 10%-20% more massive, its energy production would be dominated by the CNO cycle.

Temperature dependence of pp–chain and CNO–cycle



3.2.7 Burning of elements heavier than Hydrogen

At the beginning of their life nearly all stars burn hydrogen to helium in their cores. This ends when the hydrogen in the core is (almost) used up. Then the pressure decreases and the core contracts. For stars with $M > 0.5M_{\Theta}$ the temperature rises to ~ 10^8 K and **helium burning** starts:

$$3\,{}^{4}He \mapsto {}^{12}C + \gamma \qquad \text{triple-}\alpha \text{ process}$$

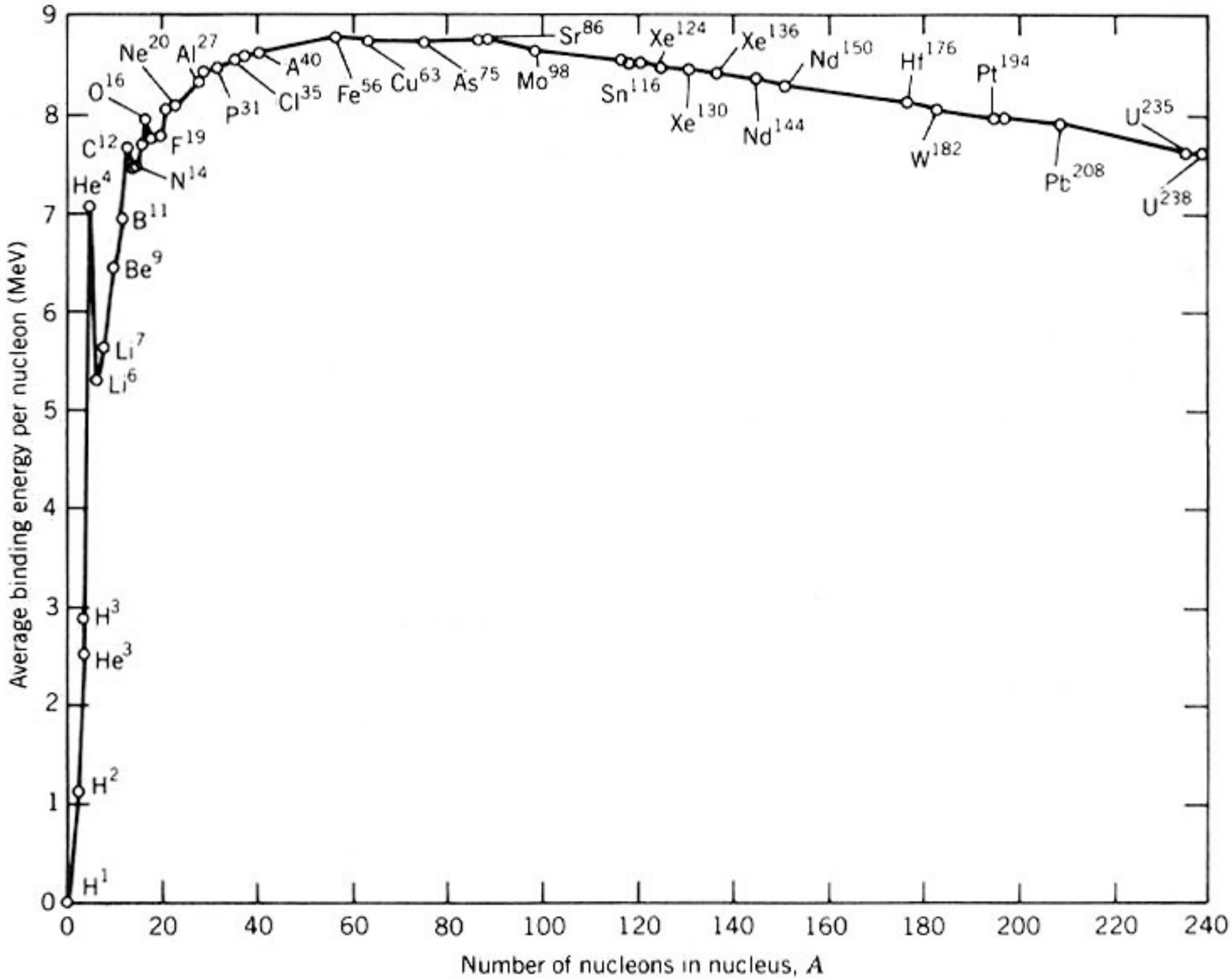
with an energy gain of 7.2 MeV. (For stars with $M < 0.5M_{\Theta}$, degeneracy of the electrons stops the contraction before higher burning can set in.) The triple-α process requires the production of ^{8}Be as an intermediate step:

$$^{4}He + {}^{4}He + 95keV \mapsto {}^{8}Be + \gamma$$

The production of ^{8}Be is endothermic, that means that ^{12}C is only produced if another ^{4}He is available "immediately":

$$^{8}Be + {}^{4}He \mapsto {}^{12}C + \gamma + 7.28MeV$$

Nuclear Binding Energy per Nucleon



Besides high temperature, helium burning requires a high enough 4He-density (> 100g/cm^3, therefore this process does not work in the big bang...). The temperature dependence of the triple-α process is:

$$\varepsilon_{3\alpha} \approx \rho^2 T^{30} (!)$$

In addition, the following reactions occur during helium burning:

$$\underbrace{{}^{12}C(\alpha,\gamma)\,{}^{16}O}_{\text{frequently}}\underbrace{(\alpha,\gamma)\,{}^{20}Ne(\alpha,\gamma)\,{}^{24}Mg(\alpha,\gamma)\,{}^{28}Si}_{\text{rarely}}$$

This explains why the cosmic abundance of C and O is the highest after H and He, and why all further burning phases rely on the fusion of C and O.

For temperatures slightly above 10^{8}K we have the following reactions:

$${}^{14}N(\alpha,\gamma)\,{}^{18}F(e^+\nu_e)\,{}^{18}O(\alpha,\gamma)\,{}^{22}Ne$$

as:

$${}^{22}Ne(\alpha,n)\,{}^{25}Mg$$

The last reaction is important, because it produces free **neutrons for the synthesis of elements with A > 60**. These cannot be produced by the capture of charged particles because their Coulomb wall is too high. The synthesis of heavy elements through slow neutron capture is called s–process (s for “slow”, rapid neutron capture only occurs in Supernova explosions).

For temperatures above T >6 × 10^8K **carbon burning** starts (only in massive stars!):

$$^{12}C+{}^{12}C\quad\begin{cases}{}^{23}Na+H & \qquad {}^{23}Na(p,\alpha)^{20}Ne\\ {}^{20}Ne+{}^{4}He & \\ {}^{23}Mg+n & \\ {}^{24}Mg+\gamma & \end{cases}$$

with

$$\varepsilon\propto\rho T^{32}$$

This is followed by **neon burning** at temperatures of T >10^9K:

$$^{20}Ne+\gamma\leftrightarrow{}^{16}O+\alpha$$

(equilibrium between production and photo-desintegration). Note: At these temperatures there already exists a background of e^+e^- pairs. The ^{16}O reacts like:

$$^{16}O(\alpha,\gamma)^{20}Ne(\alpha,\gamma)^{24}Mg(\alpha,\gamma)^{28}Si$$

At T ~ 2 × 10⁹ K **oxygen burning** sets in:

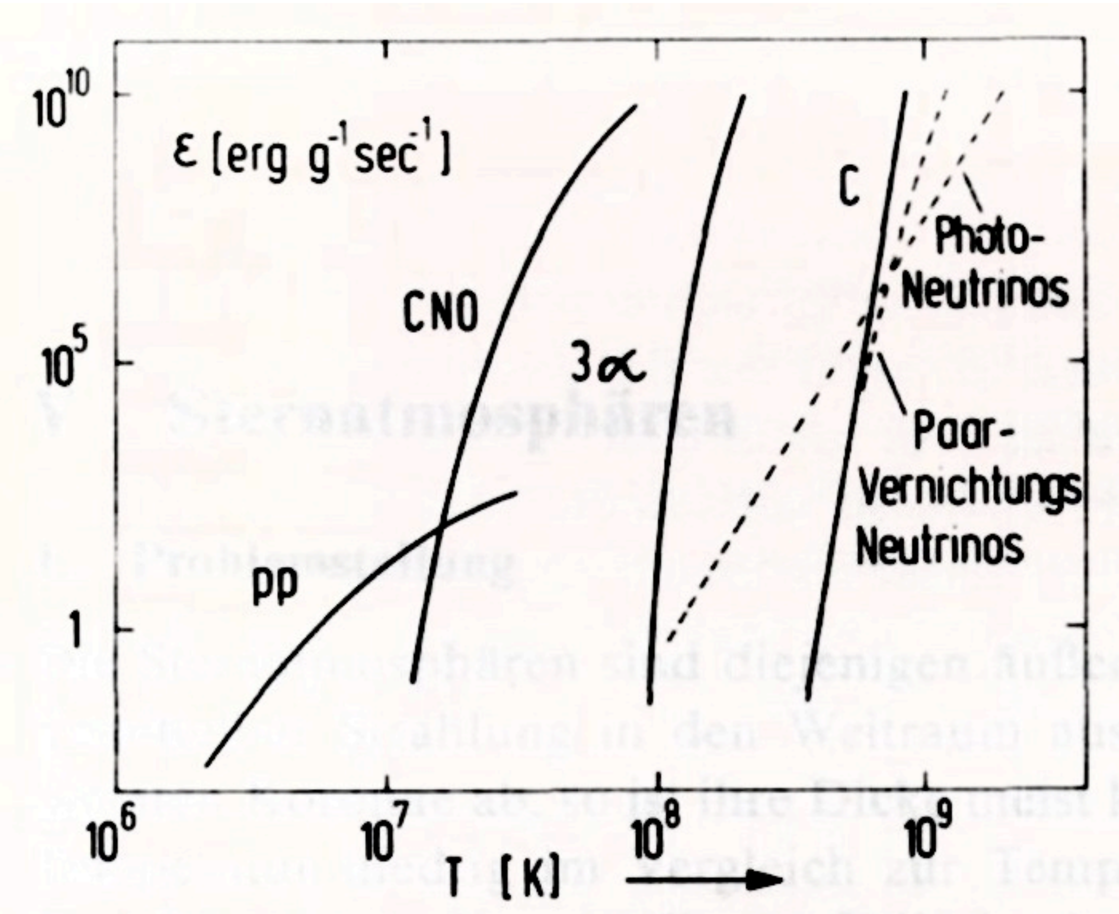
$$^{16}O + {}^{16}O \quad \begin{cases} {}^{32}S + \gamma \\ {}^{31}P + H \\ {}^{31}S + n \\ {}^{28}Si + {}^{4}He \\ {}^{24}Mg + 2\,{}^{4}He \end{cases}$$

The last burning stage is reached with **Si-burning** at T ~ 4 × 10⁹ K which **produces elements up to iron**:

$$^{28}Si \qquad {}^{56}Ni \qquad {}^{56}Co \qquad {}^{56}Fe$$

Because the binding energy per nucleus is highest for Fe, the possibilty of massive stars to generate energy ends here. The final consequence is the collapse of their cores into neutron stars or black holes which leads to a supernova explosion (see below). In the explosion but also at earlier stages, heavier elements than Fe can be built via neutron capture and *β−decay*. If the capture of neutrons is rapid compared to β-decay one speaks of **r-process** (only possible during explosive burning stages), otherwise of **s-process**. See discussion in next chapter.

Low mass stars do not reach the stage of Fe burning because their evolution stops after C and O have been synthesized. They become white dwarfs stabilized through their degenerate electron gas. See discussion in next chapter.



Energy production in stellar interiors. A density of 10^4 g/cm^{-3} was assumed for hydrogen and helium burning, 10^5 g/cm^{-3} for all other cases. The dashed lines are energy loss rates through neutrinos produced in photoncollisions and e^+e^- annihilation. (Scheffler/Elsässer: Physics of the sun and the stars).

3.3 Energy Transport

In general:

$$\left.\begin{matrix}\text{low}\\\text{high}\end{matrix}\right\}\text{ energy transport leads to }\left\{\begin{matrix}\text{steep}\\\text{flat}\end{matrix}\right\}\text{ temperature gradients and }\left\{\begin{matrix}\text{slow}\\\text{fast}\end{matrix}\right\}\text{ cooling.}$$

For stars we have two principal possibilities for the energy transport (the usual heat conduction, e.g. as in metals, is almost irrelevant in stars):

- radiation (barbecue)
- convection (earth's atmosphere)

3.3.1 Radiative energy transport

Because of the opacity of the stellar matter the energy transport via radiation takes place via photon dissipation. The mean free path of the photons l_γ corresponds to an optical depth for absorption of $\tau = \kappa l = 1$ with $\kappa = \kappa(\rho,T)$ being the absorption coefficient.

The main contributors to the opacity in the inner parts of a star are the free–free and free–bound transitions (as most atoms are ionised) and scattering of photons at free electrons. Under these conditions a reasonably good approximation for the mean opacity is the **Rosseland–opacity**:

$$\kappa \approx \sigma_T n_e \approx 10^{-4}\, cm^{-1}$$

where the Thomson cross-section of free electrons (σ_T) is:

$$\sigma_T = \frac{8\pi e^4}{3 m_e^2 c^4} = 6.6652 \times 10^{-25}\, cm^2$$

For the typical conditions of solar type stars we have

$$l_\gamma \approx 10^{-7} R_\odot$$

(Note that this is the reason for local thermodynamical equilibrium). As $l_\gamma \ll R_\odot$ we can use the diffusion approximation; the universal diffusion equation for any particle is (Fick's law, follows from simple book-keeping argument):

$$j \;=\; -\frac{1}{3}\, v_p\, l_{free}\, \frac{dn_p}{dr}$$

In our case:

- j : diffusion flux $\quad \dfrac{L(r)}{4\pi r^2 h\nu} = \text{photon flux}$

- v_p : average particle velocity $\qquad c$
- l_{free} : mean free path $\qquad l_\gamma = \frac{1}{\kappa}$
- n : photon density $\qquad n = \frac{\sigma_B T^4}{ch\nu}$

It follows:

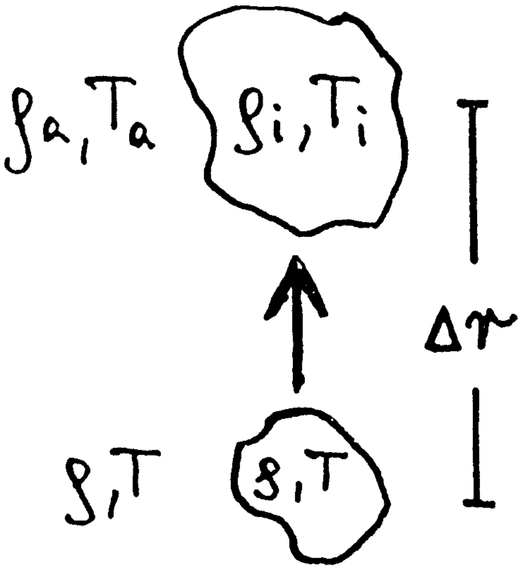
$$\frac{L(r)}{4\pi r^2} = -\frac{1}{3} c \frac{1}{\kappa} \frac{\sigma_B}{c} \frac{dT^4(r)}{dr}$$

or:

$$\frac{dT}{dr} = -\frac{3}{16\pi} \frac{\kappa}{\sigma_B T^3} \frac{L(r)}{r^2} \qquad \text{(radiation)}$$

As expected, dT/dr increases if κ or L/r^2 increases. The exact solution of the last equation requires the exact knowledge of κ which can be calculated for a given matter composition as a function of ρ and T.

3.3.2 Convective energy transport:



Consider a mass element Δm which moves upwards: At the starting point it has the same ρ and T as the surroundings. After an ascent by Δr: $\rho \rightarrow \rho_i$, $T \rightarrow T_i$. This can be different from the values of the surroundings ρ_a, T_a. If $\rho_i > \rho_a$ the mass element falls back (stable stratification), if $\rho_i < \rho_a$ buoyency sets in and the element continues to rise (→ convection). Therefore, the gradient in ρ_i needs to be larger than the gradient in ρ_a for convection to start:

$$\text{Convection if} \quad \left|\frac{d\rho}{dr}\right|_i > \left|\frac{d\rho}{dr}\right|_a$$

How does the density of the mass element change? As we can assume that the element starts to rise slowly, i.e. ($v \ll v_{sound}$), we will have pressure equilibrium, i.e. $P_i = P_a$. Assuming futhermore that the mass element is optically thick (as in case of large scale convection), no significant heat transport between the mass element and the surrounding takes place. Consequently, the process is adiabatic (may not be true at the surface of the star anymore). Thus:

$$P_i = P_a \qquad \frac{k_B}{\mu m_p}\rho_i T_i = \frac{k_B}{\mu m_p}\rho_a T_a \qquad \rho_a T_a = \rho_i T_i$$

Or:

$$\text{Convection if} \qquad \left|\frac{dT}{dr}\right|_i < \left|\frac{dT}{dr}\right|_a$$

We obtain a dimensionless form via multiplying with $\frac{P}{T}\frac{dr}{dP}$. This is the so-called **Schwarzschild–criterion:**

$$\text{Convection if} \qquad \left|\frac{d\ln T}{d\ln P}\right|_i < \left|\frac{d\ln T}{d\ln P}\right|_a$$

For an adiabatic process we have:

$$P \propto \rho^{\gamma}, \quad T \propto \rho^{\gamma-1}$$

$$\rightarrow \quad dP \propto \gamma\rho^{\gamma-1} d\rho, \quad dT \propto (\gamma-1)\rho^{\gamma-2} d\rho$$

$$\rightarrow \quad \frac{d\ln T}{d\ln\rho} = \frac{\rho^{\gamma}\rho^{\gamma-2}}{\rho^{\gamma-1}\rho^{\gamma-1}} \cdot \frac{\gamma-1}{\gamma} = 1 - \frac{1}{\gamma}$$

Using this we obtain:

$$\left|\frac{d\ln T}{d\ln P}\right|_{i} = 1 - \frac{1}{\gamma}$$

with $\gamma = c_P/c_V$ = the ratio of the specific heats at constant pressure or volume. For a one-atomic gas $\gamma = 5/3$.

$$\frac{d\ln T}{d\ln\rho} = 0.4$$

If radiative energy transport results in $\left|\frac{d\ln T}{d\ln P}\right|_{rad} > 0.4$, then convection sets in and adjusts $\left|\frac{d\ln T}{d\ln P}\right|_{con} = 0.4$. Vice versa, if $\left|\frac{d\ln T}{d\ln P}\right|_{rad} < 0.4$, convection stops (0.4 valid only for oneatomic gas).

As $c_P - c_V = k_B/\mu m_p$ we also have $1 - 1/\gamma = \kappa/\mu m_p c_P$, that means a higher specific heat leads to smaller adiabatic temperature gradients and makes convection more probable.

In regions of recombination or ionisation c_P is higher (more particles at ~ same temperature) and therefore recombination zones are mostly convective.

In a recombination zone the temperature of a rising mass element changes only slightly as the recombination releases photons which reduce the cooling through adiabatic expansion. Thus $\left|\frac{d\ln T}{d\ln P}\right|_i$ is small, which induces convection.

The depth of convection zones can only be calculated in costly numerical hydrodynamic simulations or parameterized with a simple ansatz in analogy to the mixing-length theory of Prandtl&Biermann(1925):

$$l_{\text{mixing length}} = \alpha \times R_*$$

3.4 The basic equations of stellar structure: summary

$$\frac{dP(r)}{dr} = -\rho(r) G \frac{M(r)}{r^2} \quad (3.1)$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \quad (3.2)$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r) \varepsilon(\rho, T) \quad (3.3)$$

$$\frac{dT(r)}{dr} = -\frac{3}{16\pi} \frac{\kappa}{\sigma_B T^3} \frac{L(r)}{r^2} \quad \text{radiation (3.4a)}$$

$$\frac{dT(r)}{dr} = -\frac{\mu m_p}{\kappa} \left(1 - \frac{1}{\gamma(\rho, T)} \right) G \frac{M(R)}{r^2} \quad \text{convection (3.4b)}$$

where (3.4b) follows from $\frac{d \ln T}{d \ln P} = 1 - \frac{1}{\gamma}$ and (3.1). This is a non-linear set of equations, which can be solved numerically.

In addition to (3.1) to (3.4) we have the equation of state and the formula of energy production:

$$P = \frac{\kappa}{\mu m_p} \rho T \qquad \text{ideal gas}$$

$$\varepsilon = \varepsilon_0 X_H^2 \rho f(T) \qquad \text{energy production}$$

where $X_H(r)$ is the hydrogen fraction in mass, $\mu(\rho, T, r)$ is the mean atomic weigth and f(T) is given via, e.g.:

$$f_{pp} \propto T^5 \quad \text{for} \quad T < 1.8 \cdot 10^7 K$$

$$f_{CNO} \propto T^{17} \quad \text{for} \quad T > 1.8 \cdot 10^7 K$$

3.4.1 The solution of the structure equations for the sun

Standard Solar Model [Bahcall & Pinsonneault, Phys. Lett. B, 433 (1998) 1-8]

Columns in the Standard Model table (below) represent:

1) Mass fraction in units of the solar mass
2) Radius of the zone in units of one solar radius
3) Temperature in units of deg (K)
4) Density in units of g/cm^3
5) Pressure in units of dyn/cm^2
6) Luminosity fraction in units of the solar luminosity
7) X(1H): the hydrogen mass fraction
8) X(4He): the helium 4 mass fraction
9) X(3He): the helium 3 mass fraction

mass	radius	temp.	density	pressure	luminos.	X(1H)	X(4He)	X(3He)	X(12C)	X(14N)	X(16O)
0.0000899	0.00942	1.567E+07	1.515E+02	2.321E+17	0.00081	0.34017	0.63864	7.48E-06	2.61E-05	5.93E-03	9.56E-03
0.0008204	0.01976	1.557E+07	1.480E+02	2.274E+17	0.00728	0.34981	0.62902	8.18E-06	2.54E-05	5.86E-03	9.63E-03
0.0024774	0.02875	1.542E+07	1.432E+02	2.211E+17	0.02148	0.36311	0.61574	9.22E-06	2.46E-05	5.77E-03	9.72E-03
0.0074817	0.04213	1.512E+07	1.340E+02	2.083E+17	0.06189	0.38972	0.58917	1.17E-05	2.30E-05	5.62E-03	9.85E-03
0.0214764	0.06146	1.454E+07	1.183E+02	1.851E+17	0.16215	0.43748	0.54151	1.76E-05	2.07E-05	5.45E-03	9.98E-03
0.0412353	0.07852	1.393E+07	1.040E+02	1.627E+17	0.28212	0.48312	0.49596	2.61E-05	1.87E-05	5.35E-03	1.00E-02
0.0649220	0.09385	1.334E+07	9.180E+01	1.426E+17	0.40211	0.52320	0.45596	3.77E-05	1.70E-05	5.31E-03	1.00E-02
0.0939286	0.10922	1.274E+07	8.060E+01	1.234E+17	0.52214	0.56017	0.41907	5.49E-05	1.53E-05	5.27E-03	9.97E-03
0.1309751	0.12601	1.207E+07	6.963E+01	1.040E+17	0.64221	0.59535	0.38396	8.29E-05	1.36E-05	5.25E-03	9.93E-03
0.1821073	0.14640	1.128E+07	5.804E+01	8.322E+16	0.76229	0.62976	0.34960	1.37E-04	1.05E-04	5.08E-03	9.88E-03
0.2653109	0.17616	1.019E+07	4.405E+01	5.858E+16	0.88233	0.66486	0.31448	2.87E-04	2.26E-03	1.99E-03	9.81E-03
0.3924312	0.21910	8.803E+06	2.886E+01	3.378E+16	0.96464	0.69129	0.28766	8.36E-04	3.41E-03	1.08E-03	9.74E-03
0.5077819	0.25967	7.707E+06	1.883E+01	1.941E+16	0.99136	0.70158	0.27599	2.34E-03	3.44E-03	1.07E-03	9.68E-03
0.6069345	0.29903	6.818E+06	1.221E+01	1.115E+16	0.99853	0.70612	0.27127	2.61E-03	3.43E-03	1.06E-03	9.63E-03
0.6897996	0.33815	6.075E+06	7.863E+00	6.412E+15	0.99972	0.70917	0.26983	1.07E-03	3.42E-03	1.06E-03	9.60E-03
0.7577972	0.37771	5.439E+06	5.045E+00	3.686E+15	1.00001	0.71108	0.26862	4.47E-04	3.41E-03	1.06E-03	9.57E-03
0.8128569	0.41816	4.884E+06	3.228E+00	2.119E+15	1.00007	0.71248	0.26749	2.22E-04	3.40E-03	1.05E-03	9.54E-03
0.8569686	0.45981	4.392E+06	2.062E+00	1.218E+15	1.00007	0.71369	0.26642	1.42E-04	3.39E-03	1.05E-03	9.51E-03
0.8919826	0.50286	3.951E+06	1.316E+00	7.004E+14	1.00006	0.71481	0.26537	1.14E-04	3.38E-03	1.05E-03	9.49E-03
0.9195303	0.54740	3.550E+06	8.416E-01	4.027E+14	1.00004	0.71589	0.26433	1.05E-04	3.37E-03	1.05E-03	9.47E-03
0.9410043	0.59333	3.176E+06	5.403E-01	2.316E+14	1.00003	0.71678	0.26345	1.01E-04	3.37E-03	1.05E-03	9.47E-03
0.9575656	0.64028	2.815E+06	3.503E-01	1.331E+14	1.00002	0.71763	0.26227	1.00E-04	3.43E-03	1.06E-03	9.63E-03
0.9701625	0.68738	2.432E+06	2.317E-01	7.656E+13	1.00002	0.72679	0.25375	1.00E-04	3.32E-03	1.03E-03	9.32E-03
0.9738342	0.70387	2.282E+06	2.018E-01	6.290E+13	1.00001	0.73331	0.24792	1.00E-04	3.20E-03	9.93E-04	8.99E-03
0.9782230	0.72563	2.060E+06	1.706E-01	4.817E+13	1.00001	0.73902	0.24276	1.00E-04	3.11E-03	9.64E-04	8.73E-03
0.9821147	0.74708	1.846E+06	1.445E-01	3.653E+13	1.00001	0.73902	0.24276	1.00E-04	3.11E-03	9.64E-04	8.73E-03
0.9881550	0.78631	1.483E+06	1.037E-01	2.101E+13	1.00001	0.73902	0.24276	1.00E-04	3.11E-03	9.64E-04	8.73E-03
0.9923273	0.82067	1.191E+06	7.437E-02	1.209E+13	1.00001	0.73902	0.24276	1.00E-04	3.11E-03	9.64E-04	8.73E-03
0.9951255	0.85036	9.572E+05	5.337E-02	6.954E+12	1.00001	0.73902	0.24276	1.00E-04	3.11E-03	9.64E-04	8.73E-03
0.9969548	0.87569	7.691E+05	3.831E-02	4.000E+12	1.00001	0.73902	0.24276	1.00E-04	3.11E-03	9.64E-04	8.73E-03
0.9981246	0.89708	6.181E+05	2.750E-02	2.302E+12	1.0000[illegible]	0.73902	0.24276	1.00E-04	3.11E-03	9.64E-04	8.73E-03
0.9988589	0.91497	4.970E+05	1.974E-02	1.324E+12	1.00000	0.73902	0.24276	1.00E-04	3.11E-03	9.64E-04	8.73E-03

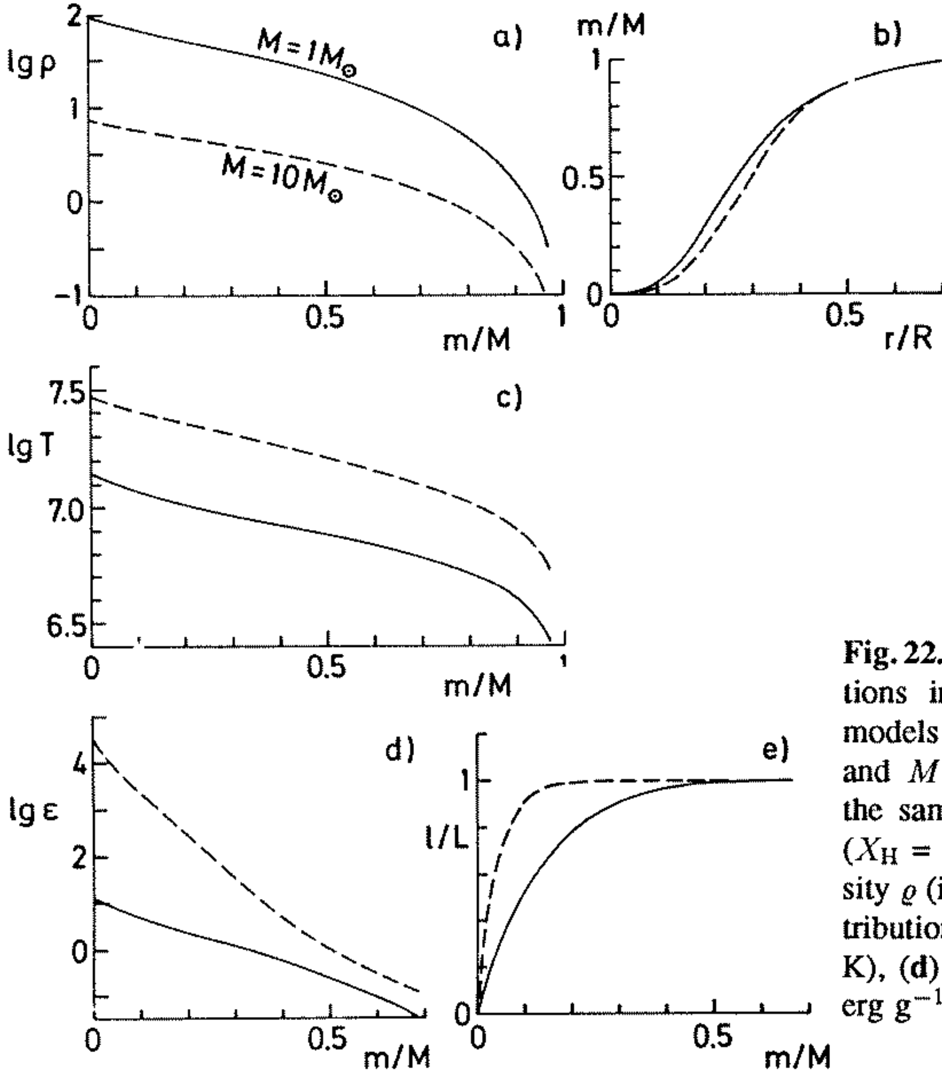
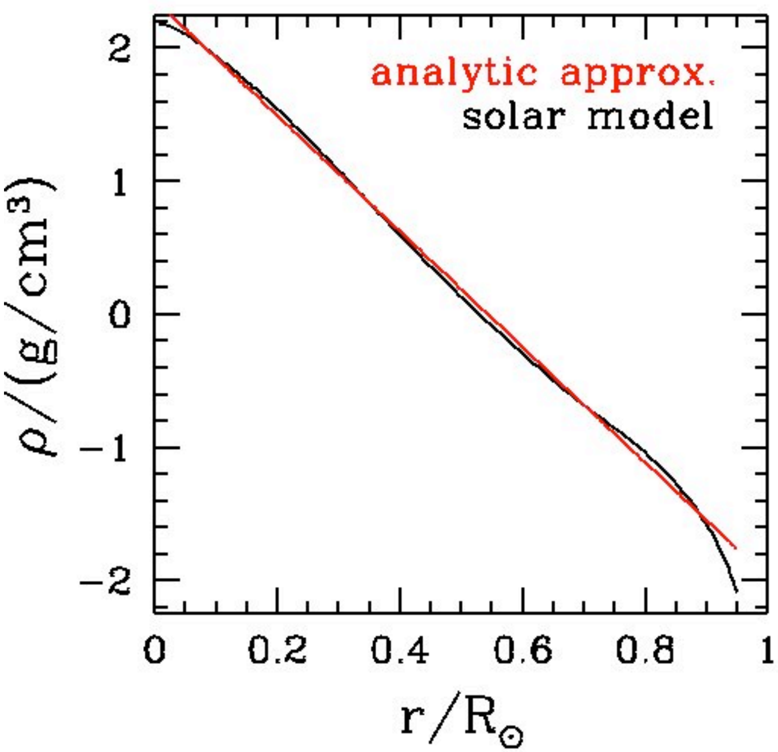


Fig. 22.4. a, b The run of some functions inside zero-age main-sequence models for $M = 1M_\odot$ (*solid lines*) and $M = 10M_\odot$ (*dashed lines*) with the same composition as in Fig. 22.1 ($X_\mathrm{H} = 0.685$, $X_\mathrm{He} = 0.294$); (**a**) density ϱ (in g cm^{-3}), (**b**) radial mass distribution $m(r)$, (**c**) temperature T (in K), (**d**) nuclear energy production (in erg g^{-1} s^{-1}), (**e**) local luminosity l

Diagram from Kippenhahn & Weigert, Stellar Structure and Evolution.

3.5 An analytical model for main sequence stars

Correct solutions for the main sequence can only be obtained with numerical methods. However, a simple but pretty good analytical model which provides important physical Insights is the following:



- Main sequence stars are chemically homogeneous.

- The density profile is parameterized by:

$$\rho(r) = \rho_c e^{-r/H} \qquad (3.5)$$

  with H = δ·R, R =radius of star, and δ~ 0.1. For the sun ρ_c = 230 g/cm^3.

- The energy production is constrained to r/R < 2δ:

$$\varepsilon(\mathrm{r}) = \begin{cases} \varepsilon_c = \varepsilon_0 \rho_c T_c^{\alpha} & \text{for } r < 2\delta R \\ 0 & \text{for } r > 2\delta R \end{cases} \qquad (3.6)$$

The standard model of the sun and other main sequence stars shows that these approximations are o.k. With (3.5) and (3.6) and the basic equations (3.1)-(3.4) it is possible to derive the scaling relations of the main sequence. Integration of (3.2) via inserting of (3.5) yields:

$$M(r) = 4\pi\rho_c \int_0^r r'^2 \, e^{-r'/H} dr'$$

$$M(r) = 4\pi\rho_c 2H^3 \left[1 - e^{r/H}\left(1 + \frac{r}{H} + \frac{1}{2}\frac{r^2}{H^2}\right)\right] \tag{3.7}$$

For r → R we have $e^{-r/H} \rightarrow e^{-1/\delta} \sim 0$, which means:

$$M(R) = 4\pi\rho_c 2H^3$$

or

$$\rho_c = \frac{1}{6\delta^3}\frac{M}{4\pi/3R^3} = \frac{1}{6\delta^3}\overline{\rho} \sim 170\overline{\rho} \tag{3.8}$$

For the pressure we derive from (3.1) and (3.7):

$$P(R) - P_c = -G\int_0^R \frac{M(r)}{r^2}\rho(r)dr$$

or

$$P_c = G4\pi\rho_c^2 2H^3 \int_0^R \frac{e^{-r/H}}{r^2}\left[1 - e^{-r/H}\left(1 + \frac{r}{H} + \frac{1}{2}\frac{r^2}{H^2}\right)\right]dr$$

Setting x = r/H it follows:

$$P_c = G4\pi\rho_c^2 2H^2 \underbrace{\int_0^{1/\delta} \frac{e^{-x}}{x^2}\left[1-e^{-x}\left(1+x+\frac{1}{2}x^2\right)\right]}_{\approx 5.6\cdot 10^{-2} \quad \text{for} \quad 1/\delta \gg 1}$$

and thus:

or

$$P_c = 5.6\times 10^{-2}\, G\frac{4\pi}{3}\rho_c^2 6\delta^2 R^2 \tag{3.9a}$$

$$P_c = 2.2\times 10^{-3}\frac{GM^2}{R^4}\times\frac{1}{\delta^4} \tag{3.9b}$$

and using the ideal gas law $P_c = k_B/\mu_c m_p \cdot \rho_c T_c$ and omitting radiation pressure, we obtain the equivalent of the Virial Equation for stars:

$$T_c = 5.6\times 10^{-2}\frac{\mu_c m_p}{k_B} G\frac{M}{R}\times\frac{1}{\delta} \tag{3.10}$$

From this we conclude for main sequence stars:

higher mass
→ higher gravitation
→ higher pressure and higher temperature

For the luminosity we obtain from (3.3) and (3.6):

$$L = 4\pi X_H^2 \int_0^{2H} r^2 \rho^2 \varepsilon_0 T_c^\alpha dr$$

$$= 4\pi X_H^2 \varepsilon_0 T_c^\alpha \rho_c^2 \int_0^{2H} e^{-2r/H} r^2 dr$$

$$= \varepsilon_0 T_c^\alpha \rho_c^2 X_H^2 \frac{4\pi}{3} R^3 \delta^3 \cdot \frac{3}{4}\left[1 - e^{-4} \cdot 13\right]$$

where describes the temperature dependence of the fusion process ($\alpha = 5$ for pp and $\alpha = 17$ for CNO,...) and we have used

$$\int e^{ax} x^2 dx = e^{ax}\left(\frac{x^2}{a^2} - \frac{2x}{a^2} + \frac{2}{a^3}\right)$$

$$L = 3.8 \times 10^{-3} \frac{X_H^2 \varepsilon_0 T_c^\alpha}{\delta^3} \frac{M^2}{R^3} \qquad (3.11a)$$

$$L = 0.096 M \varepsilon_0 X_H^2 \rho_c T_c^\alpha \qquad (3.11b)$$

where the last equation was derived using (3.8). As $\varepsilon_0 X_H^2 \rho_c T_c^\alpha$ is the energy production rate per gram, (3.11b) implies that around 10% of the total mass is involved in the fusion process. Equation (3.11) does not yet fix the luminosity of a star of given mass M, as we have still the freedom to choose T_c. T_c can be determined via the energy transport equation which will finally yield the unique dependence of L on M as claimed in the Russel-Vogt theorem.

To solve the energy transport equation (3.4a) we assume that

$$T \approx T_c \quad \text{for} \quad r < 2H$$

and we solve only for r > 2H beyond which L is constant. For the opacity we make the rather wild approximation:

$$\bar{\kappa} \approx \kappa_0 \cdot \rho \quad \text{with} \quad \kappa_0 = const. \tag{3.12}$$

Furthermore, we assume that we only have radiative energy transport (cf. eq. 3.4a):

$$\frac{dT^4}{dr} = -\frac{3}{4\pi\sigma_B}\kappa_0 \rho \frac{L(r)}{r^2}$$

Integrating from 2H to R it follows (L =const!):

$$T^4(R) - T_c^4 = -\frac{3}{4\pi\sigma_B}\kappa_0 L \rho_c \int_{2H}^{R} \frac{e^{-(r/H)}}{r^2} dr$$

This integral can be solved by substituting again x = r/H:

$$\int_{2H}^{R} \frac{e^{-(r/H)}}{r^2} dr = \frac{1}{H}\int_{2}^{1/\delta} \frac{e^{-x}}{x^2} dx \approx \int_{2}^{\infty} \frac{e^{-x}}{x^2} dx = 1.9 \cdot 10^{-2} \frac{1}{H}$$

and because of $T^4(R) \ll T_c^4$ for $r > 2H$ we have:

$$T_c^4 = 1.9 \times 10^{-2} \frac{\kappa_0}{\sigma_B} L \frac{3}{4\pi} \frac{\bar{\rho}}{R} \frac{1}{6\delta^4}$$

$$T_c^4 = \frac{3.2 \times 10^{-3}}{\delta^4} \frac{\kappa_0}{\sigma_B} L \frac{M}{R^4} \qquad (3.13)$$

Equation (3.13) states how the opacity changes the central temperature for fixed L, M and R (high opacity → steep T-gradient → high central temperature). In contrast, equation (3.10) determines T_c in such way that hydrostatic equilibrium is satisfied (this corresponds to the virial theorem). Putting these equations together we finally obtain the mass-luminosity equation of the main sequence:

$$L = 3 \times 10^{-3} \left(\frac{\mu_c m_p}{k_B} \right)^4 \frac{\sigma_B}{\kappa_0} M^3 \qquad (3.14)$$

Note: for constant mass the following applies

- $L \sim \kappa_0^{-1}$ Higher opacity → smaller luminosity
- $L \sim \mu_c^4$ Fusion H to He → higher μ_c → higher L.

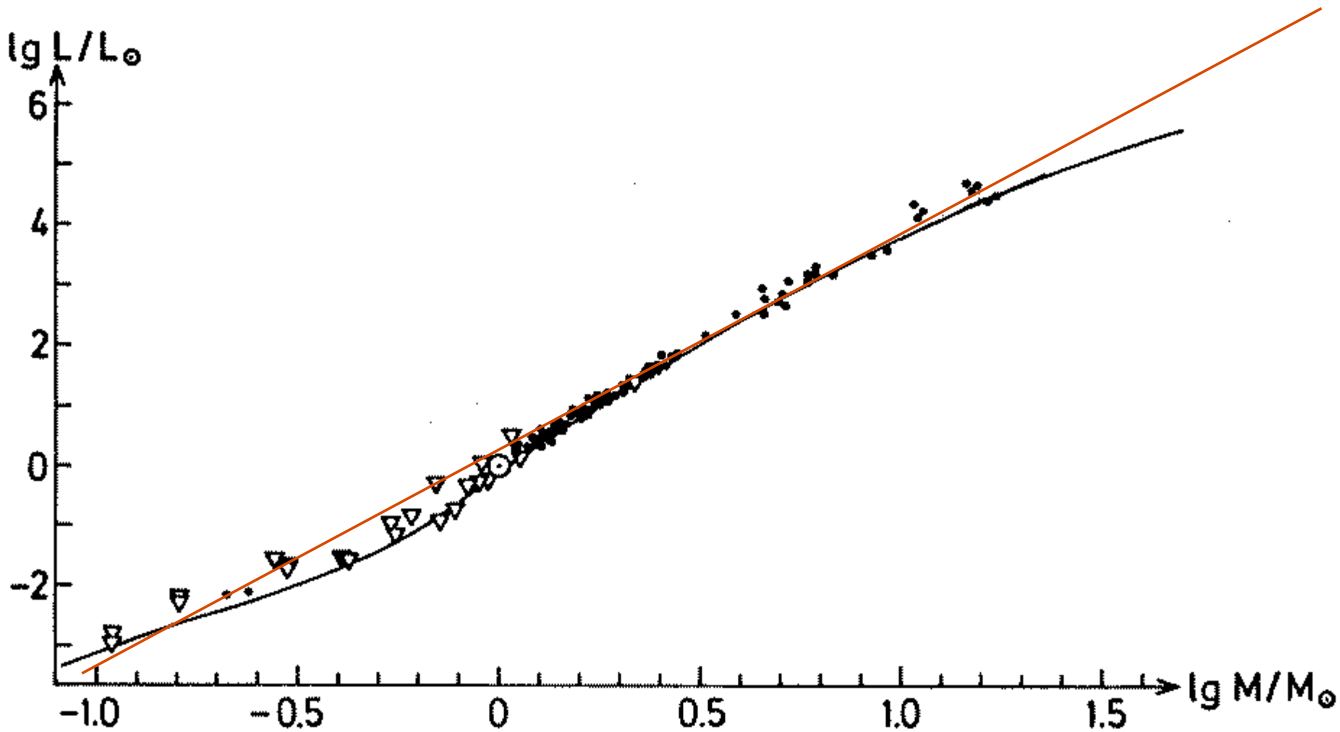


Fig. 22.3. The line gives the mass–luminosity relation for the models of the main sequence shown in Fig. 22.1. Measurements of binary systems are plotted for comparison (the symbols have the same meaning as in Fig. 22.2)

from Kippenhahn & Weigert: Stellar Structure and Evolution

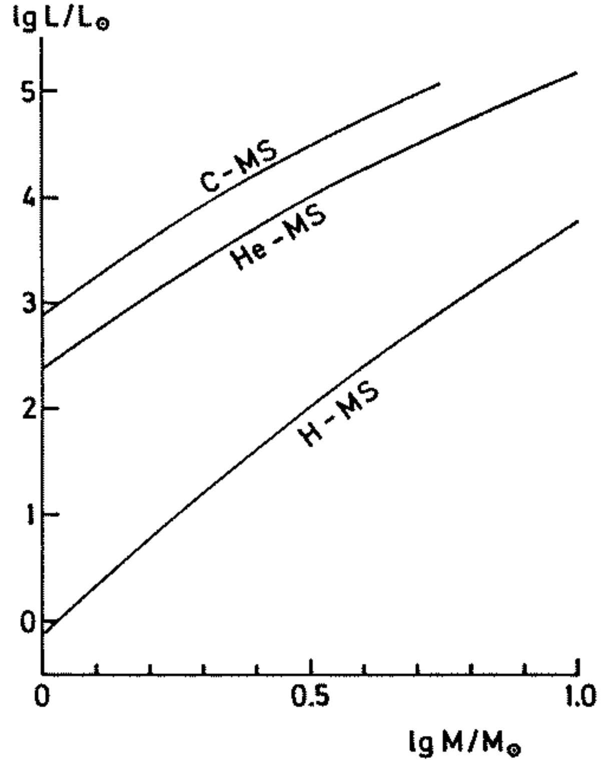


Fig. 23.2. Mass–luminosity relations for the models of the hydrogen, helium, and carbon main sequences of Fig. 23.1

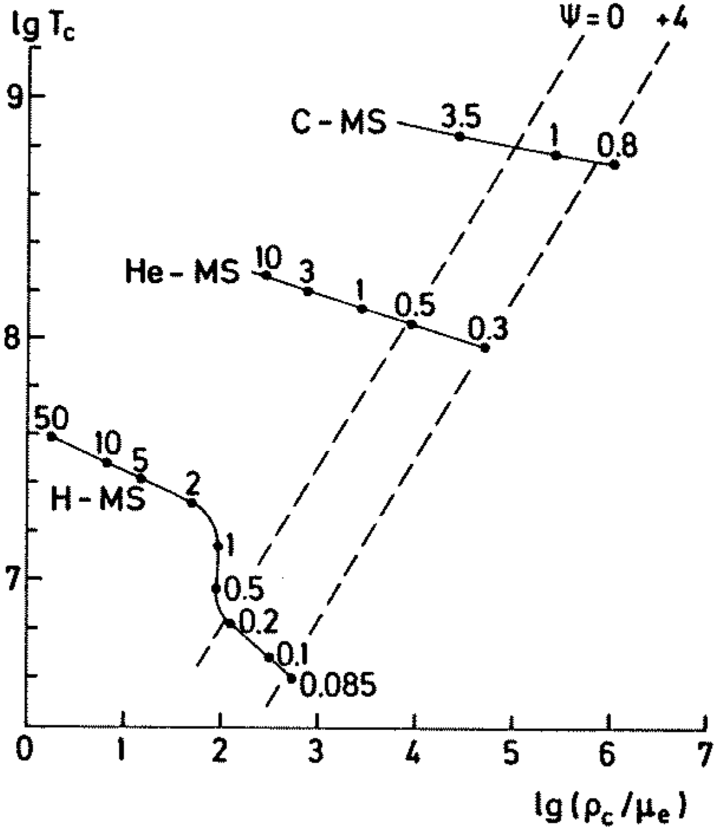


Fig. 23.3. Central temperature T_c (in K) and central density ϱ_c/μ_e (ϱ_c in g cm^{-3}, μ_e = molecular weight per electron) of the models on the hydrogen, helium and carbon main sequences of Fig. 23.1. The labels along the lines give the stellar mass M (in $M_\odot$). The dashed lines indicate constant degeneracy parameters ψ of the electron gas

Inserting of (3.13) and (3.10) into (3.11a) results in the mass–radius–relation:

$$R \propto \mu_c^{\frac{\alpha-4}{\alpha+3}} \left(X_H^2 \varepsilon_0 \kappa_0\right)^{\frac{1}{\alpha+3}} M^{\frac{\alpha-1}{\alpha+3}} \qquad (3.15)$$

Inserting yields:

- pp: α = 5: $\Rightarrow \quad R \propto \mu_c^{\frac{1}{8}} \left(X_H^2 \varepsilon_0 \kappa_0\right)^{\frac{1}{8}} M^{\frac{1}{2}}$
- CNO: α = 17: $\Rightarrow \quad R \propto \mu_c^{\frac{13}{20}} \left(X_H^2 \varepsilon_0 \kappa_0\right)^{\frac{1}{20}} M^{\frac{4}{5}}$

which agrees with the observed relation $R \propto M^{0.6}$ Inserting in (3.10) gives T_c as a function of mass:

- pp: α = 5: $\Rightarrow \quad T_c \propto \mu_c^{\frac{7}{8}} M^{0.5}$
- CNO: α = 17: $\Rightarrow \quad T_c \propto \mu_c^{\frac{7}{20}} M^{0.2}$

Note:

- T_c decreases with decreasing mass. Numerical calculations show that below $M < 0.08 M_\odot$ no nuclear H-burning can take place → brown dwarfs. For $M < 1.4 M_\odot$ → pp-chain, for $M > 1.4 M_\odot$ → CNO-cycle.
- T_c increases with μ_c; therefore the temperature rises with increasing age as H→He.

With these equations we can now determine the effective temperature as a function of the mass:

$$T_{eff}^4 \propto \frac{L}{R^2} \propto \frac{1}{R^2} \frac{\mu_c^4}{\kappa_0} M^3$$

Replacing R using (3.15) one obtains:

$$T_{eff} \propto \mu_c^{\frac{10+\alpha}{6+2\alpha}} X_H^{-\frac{1}{3+\alpha}} \kappa_0^{-\frac{5+\alpha}{12+4\alpha}} M^{\frac{11+\alpha}{12+4\alpha}} \qquad (3.16)$$

or

- pp: α = 5 : $T_{eff} \propto \mu_c^{\frac{15}{16}} X_H^{-\frac{1}{8}} \kappa_0^{-\frac{5}{16}} M^{\frac{1}{2}}$
- CNO: α = 17 : $T_{eff} \propto \mu_c^{\frac{27}{40}} X_H^{-\frac{1}{20}} \kappa_0^{-\frac{11}{40}} M^{\frac{7}{20}}$

Note that according to this:

- T_{eff} increases with increasing mass.
- T_{eff} decreases with increasing κ_0, i.e. at a given mass, metal poor stars are hotter.
- T_{eff} increases with increasing μ_c in the pp-chain and in the CNO-cycle.

Equations (3.14) and (3.16) explain an important observation in the MilkyWay: Metal poor stars in the galactic halo (population II stars) form a main sequence which is located below the main sequence of the solar neighborhood stars (population I). Halo stars have $M < 1M_{\odot}$, i.e. have the following relations:

$$T_{eff} \propto \mu_c \kappa_0^{-0.3} M^{\frac{1}{2}}$$

$$L \propto \mu_c^4 \kappa_0^{-1} M^3$$

Therefore the main sequence shifts as follows (for fixed mass):

Higher metallicity implies higher κ, ($\kappa \uparrow \rightarrow n_{e-} \uparrow \rightarrow \kappa_0 \uparrow$). Therefore, the main sequence of metal rich stars lies to the right (at lower temperature) of the main sequence of a metal poor stars. Or, put in another way, metal rich stars have a higher L and M for fixed T_{eff} .
→ This can be used to guesstimate the metal content z without spectroscopy.

The above formulae can, after taking into account the different temperature dependence of the energy production, also be used to determine the location of the He or C main sequences. These lie, at a given mass, both at higher temperature and luminosity.

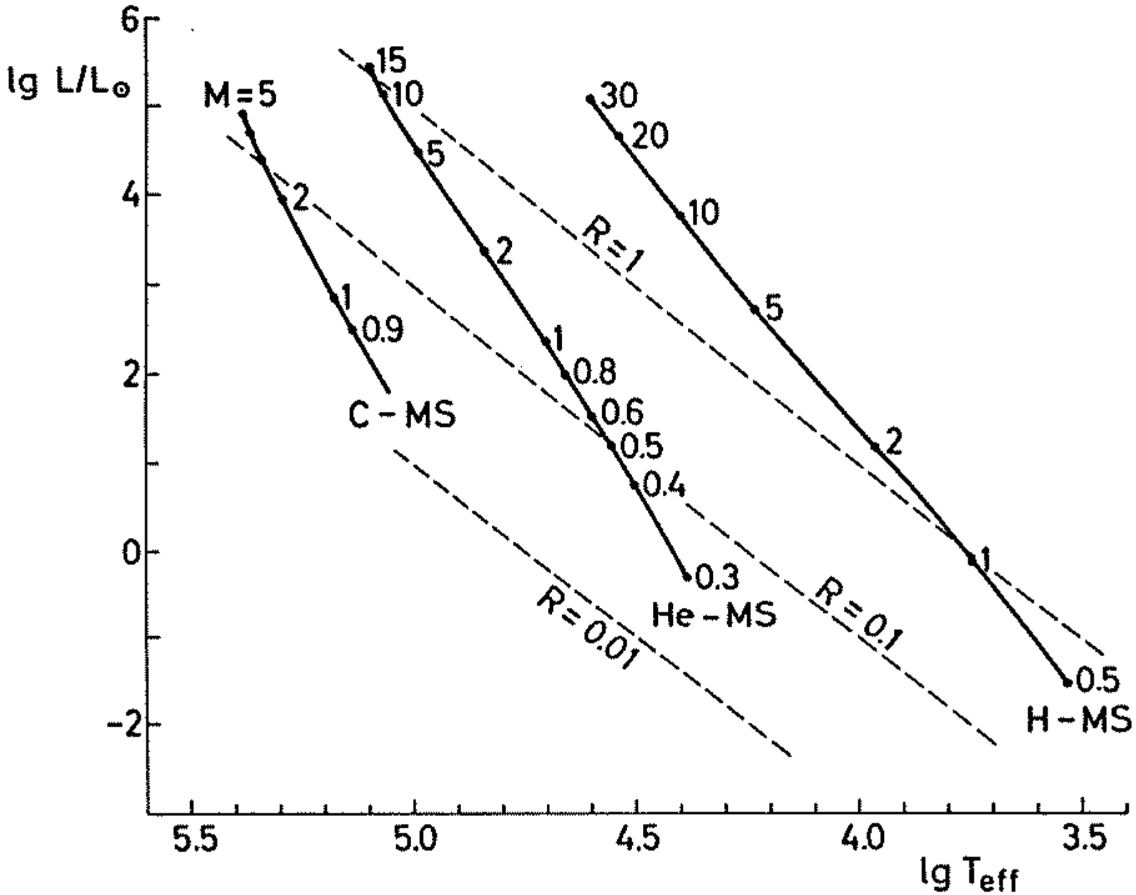


Fig. 23.1. In the Hertzsprung–Russell diagram the solid lines show the normal hydrogen main sequence (H-MS; $X_{\mathrm{H}} = 0.685$, $X_{\mathrm{He}} = 0.294$), the helium main sequence (He-MS; $X_{\mathrm{H}} = 0$, $X_{\mathrm{He}} = 0.979$) and the carbon main sequence (C-MS; $X_{\mathrm{H}} = X_{\mathrm{He}} = 0$, $X_{\mathrm{C}} = X_{\mathrm{O}} = 0.497$). The labels along the sequences give stellar masses M (in units of $M_{\odot}$). Three lines of constant stellar radius (R in units of $R_{\odot}$) are plotted (*dashed*)

Combining (3.14) and (3.16) we obtain luminosity–temperature–relation on the main sequence:

$$L \propto \mu_c^{\frac{-2\alpha-16}{11+\alpha}} X_H^{\frac{12}{11+\alpha}} \kappa_0^{\frac{4+2\alpha}{11+\alpha}} T_{eff}^{\frac{36+12\alpha}{11+\alpha}} \tag{3.17}$$

or

- pp: α = 5 : $L \propto \mu_c^{\frac{-13}{8}} X_H^{\frac{3}{4}} \kappa_0^{\frac{7}{8}} T_{eff}^{6}$
- CNO: α = 17 : $L \propto \mu_c^{\frac{-25}{14}} X_H^{\frac{3}{7}} \kappa_0^{\frac{19}{14}} T_{eff}^{\frac{60}{7}}$

So, $L \sim T_{eff}^{6...8.5}$ with is in agreement with the observation $L \sim T_{eff}^{7}$.

The fact that we can derive two-parameter relations as this one or the others above are the analytical manifestation of the **Russel-Vogt-Theorem** (but remember the comments made above).

To derive the nuclear time scales and main sequence life time, we use from eq. (3.11b):

$$L = \underbrace{0.096M}_{\text{mass used for fusion}} \times \underbrace{\varepsilon_0 X_H^2 \rho_c T_c^\alpha}_{\varepsilon_c \text{ = energy production per gram and time}}$$

The energy per gram available at the beginning is:

$$\underbrace{\frac{\Delta E}{4m_p}}_{H \quad He \text{ fusion for hydrogen gas}} \times \underbrace{\frac{3}{4}}_{\text{only 3/4 of the initial mass in hydrogen (90\% in number)}}$$

with ΔE = 26.21 MeV. The life time on the main sequence is then approximately:

$$\tau_{nuc} = \frac{\text{energy content}}{\text{energy consumption}} \approx \frac{3}{4}\frac{\Delta E}{4m_p \varepsilon_c}$$

or, using (3.11b):

$$\tau_{nuc} \approx 0.02\frac{\Delta E M}{L m_p} \approx 9 \cdot 10^9\, yr \left(\frac{M/M_\odot}{L/L_\odot}\right)$$

This can be compared to the results of a numerical model: $\tau \approx 7 \times 10^9$yr.

Using the mass–luminosity–relation $L \sim M^3$ it follows:

$$\tau_{nuc} \approx 8 \cdot 10^9\, yr \left(\frac{M}{M_\odot} \right)^{-2}$$

Note:

- the most massive stars only live < 10^8yr.
- the least massive stars live > 10^{11}yr.
- after $t \approx \tau_{nuc}$ the central hydrogen is used up, the star leaves the main sequence.
  → hydrogen shell burning
  → helium burning (dependent on mass)

The approximate analytical model is able to predict qualitatively the essential properties of main sequence stars. One key assumption of the model was radiative energy transport. Numerical models show, however, that convective zones are present in nearly all stars.

3.6 Convection, fully convective stars, Hayashi–line

Convection sets in if

$$\left|\frac{d\ln T}{d\ln P}\right|_{rad} = 1 - \frac{1}{\gamma} = \frac{k}{\mu m_H c_P}$$

with 1 − 1/γ = 0.4 for one–atomic gas and 1 − 1/γ < 0.4 in ionisation– or recombination–zones. Using that

$$\frac{d\ln T}{d\ln P} = \frac{P}{T}\frac{dT}{dP} = \frac{k}{\mu m_H}\rho\frac{dT}{dP}$$

and dividing (3.4a) by (3.1) it follows

$$\frac{3}{16}\frac{k}{4\pi\sigma_B}\frac{\bar{\kappa}}{\mu_H}\frac{1}{T^3}\frac{L(r)}{M(r)} > 1 - \frac{1}{\gamma}$$

In our analytical approximation, ignoring the density gradient, inside the nuclear burning region we have:

$$dL(r) \approx \varepsilon_0 \rho_c^2 T_c^\alpha 4\pi r^2 dr$$

$$L(r) \approx \varepsilon_0 \rho_c^2 T_c^\alpha \frac{4\pi}{3} r^3$$

$$L(r) \approx M(r)\varepsilon_0 \rho_c T_c^\alpha$$

Inserting this into the equation above yields the criterion for instability:

$$\frac{3k}{64\pi}\frac{\bar{\kappa}}{\mu_H \sigma_B}\varepsilon_0 \rho_c T_c^{\alpha-3} > 1 - \frac{1}{\gamma} = 0.4$$

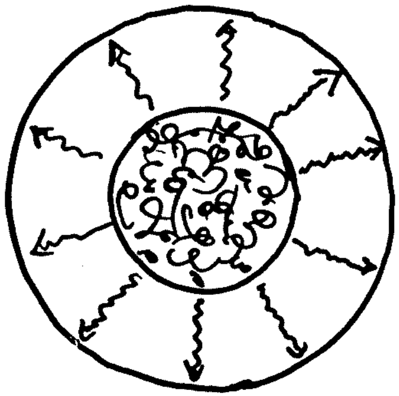
Using the above derived values for T_c and ρ_c as functions of mass gives:

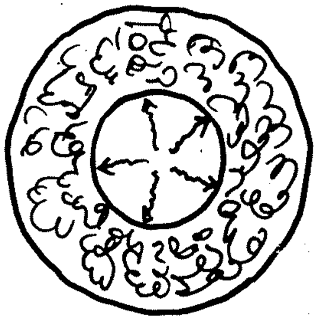
$$M \geq 1.4 M_\odot \qquad \rightarrow \qquad \text{core convective}$$
$$M \leq 1.4 M_\odot \qquad \rightarrow \qquad \text{core radiative}$$

Moreover, outside the nuclear burning zone we can also have recombination zones:

$$He^{++} \quad \leftrightarrow \quad H^{+}$$
$$He^{+} \quad \leftrightarrow \quad He$$
$$H^{+} \quad \leftrightarrow \quad H$$
$$H \quad \leftrightarrow \quad H^{-}$$

In these areas c_P is large ($1 - 1/\gamma$ small), there is almost always convection. The cooler a star, the deeper this convection reaches into the star. We find two type of main sequence stars:





$M \geq 1.4 M_{\odot}$

- hot stars
- convective core
- radiative envelope
- CNO–cycle restricted to the center
- radiation pressure important for $M \geq 20 M_{\odot}$

$M \leq 1.4 M_{\odot}$

- cool stars
- radiative core
- convective envelope
- pp–chain slightly concentrated to the center
- fully convective for $M \leq 0.3 M_{\odot}$

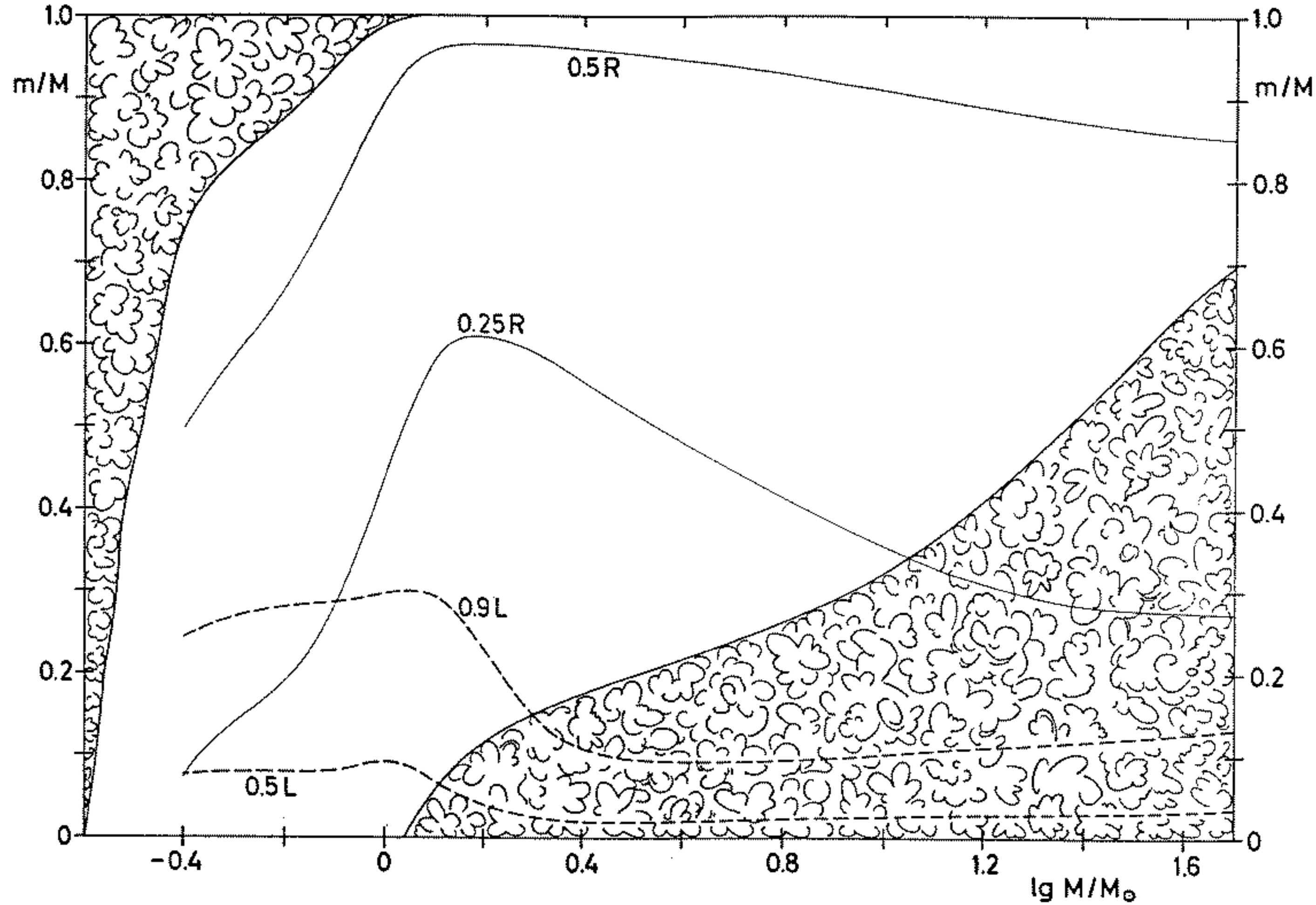


Fig. 22.7. The mass values m from centre to surface are plotted against the stellar mass M for the same zero-age main-sequence models as in Fig. 22.1. "Cloudy" areas indicate the extension of convective zones inside the models. Two solid lines give the m values at which r is 1/4 and 1/2 of the total radius R. The dashed lines show the mass elements inside which 50% and 90% of the total luminosity L are produced

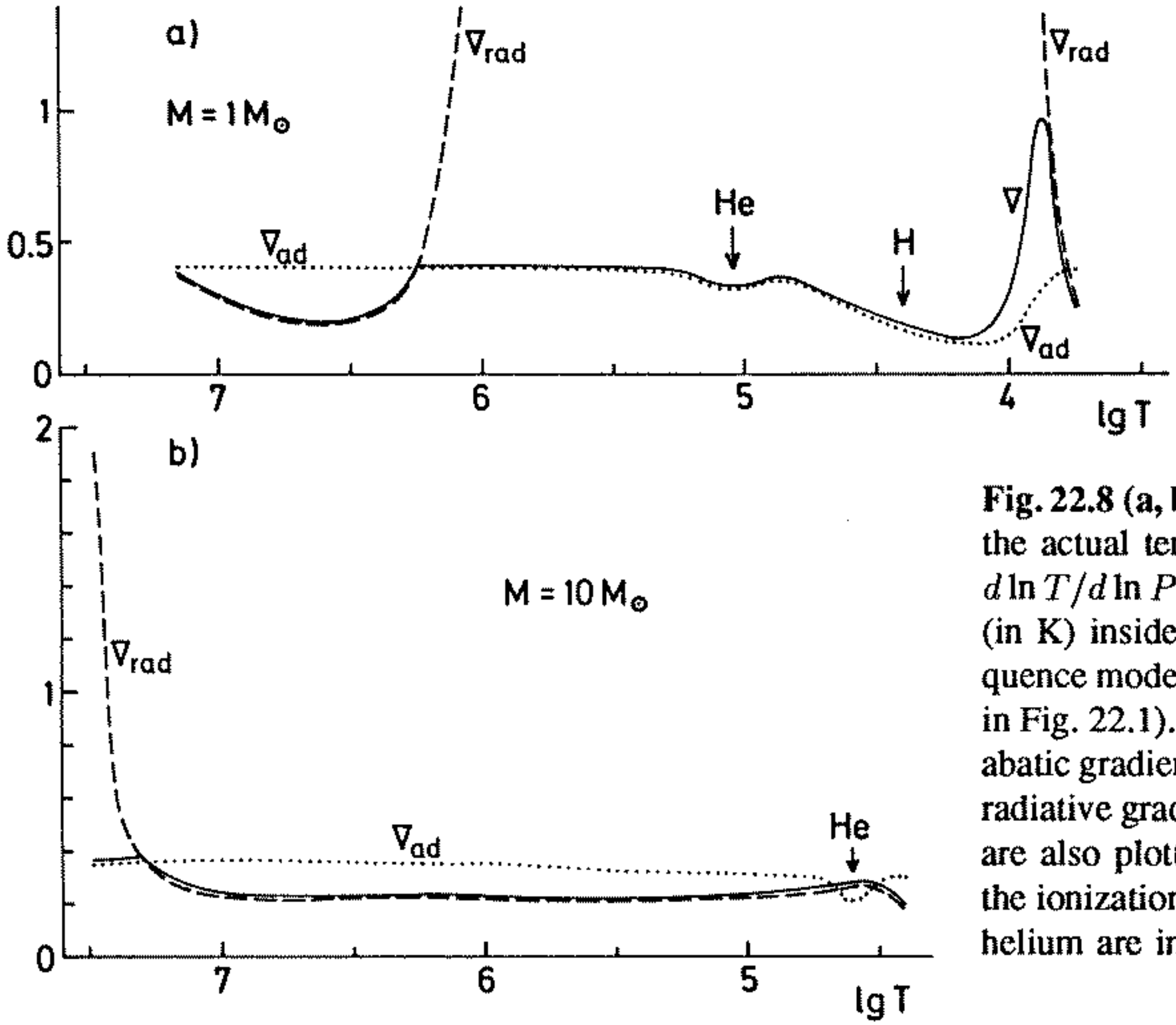


Fig. 22.8 (a, b). The solid lines show the actual temperature gradient $\nabla = d\ln T/d\ln P$ over the temperature T (in K) inside two zero-age main-sequence models (same composition as in Fig. 22.1). The corresponding adiabatic gradients ∇_{ad} (*dotted lines*) and radiative gradients ∇_{rad} (*dashed lines*) are also plotted, and the location of the ionization zones of hydrogen and helium are indicated (*arrows*)

Fully convective stars

Fully convective stars are not only found for $M < 0.3M_\odot$ on the main sequence but also stars on the giant branch as well as young stars before ignition of nuclear burning (see → star formation, → stellar evolution) are fully convective. The properties of fully convective stars were investigated in the beginning of the 60's by Hayashi.

The notation 'fully convective' is a little misleading, because the star has an outer layer with radiative energy transport (the star shines!). Literally fully convective stars would be infinitely cool and infinitely large.

For the understanding of fully convective stars linking the conditions of the central regions with the stellar atmosphere is essential. Here we present an approximation to the exact calculation.

In the inner parts of fully convective stars, the adiabatic relation between pressure and temperature applies:

$$P = CT^{\frac{\gamma}{\gamma-1}}$$

where C is constant, C = C(M,R,μ). C is constant because for each radius an ascending adiabatic mass element always has to be in P-, T- equilibrium with the environment.

To determine the effective temperature of fully convective stars requires that we determine the pressure in the stellar atmosphere. In the stellar atmosphere, the photons escape to infinity on average from the radius where the optical depth is unity (τ = 1). The theory of stellar atmospheres links the opacity with the gravitaional acceleration and the Temperature via:

$$\bar{\kappa}(\tau = 1) = \frac{g(\tau = 1)\mu m_p}{kT(\tau = 1)}$$

For $\bar{\kappa}$ we have in the sun in cgs units:

$$\bar{\kappa} \approx 6.9 \cdot 10^{-26} \rho P^{0.7} T^{5.3}$$

Inserting and using $\frac{kT}{\mu m_p} = \frac{P}{\rho}$ yields:

$$6.9 \cdot 10^{-26} P^{1.7} T^{5.3} \approx g$$

With $g = 2.74 \frac{(M / M_\odot)}{(R / R_\odot)^2}$ we get:

$$P \propto T^{-3.1} M^{0.6} R^{-1.2} \qquad \text{for } \tau = 1$$

This is the pressure for τ = 1, from which photons escape on average, i.e. at the radius where convective energy transport changes to radiative energy transport. This pressure must equal the pressure resulting from convective energy transport from the center:

$$P_{\tau=1} = P_c = C T_{eff}^{\frac{\gamma}{\gamma-1}}$$

where the temperature in the transition area was set to T_{eff}. Using the approximation γ = 5/3 we have

$$CT_{eff}^{\frac{5}{2}} \propto T_{eff}^{-3.1} M^{0.6} R^{-1.2} \qquad (*)$$

In this equation, C still is a function of M,R and μ, and can be determined from the parameters in the core:

$$P_c = CT_c^{2.5}$$

From the analytical model we get (Eqs. 3.9b and 3.10):

$$P_c \propto \frac{M^2}{R^4}$$

$$T_c \propto \mu \frac{M}{R}$$

With this we get for C:

$$C \propto M^{-0.5} R^{-1.5} \mu^{-2.5}$$

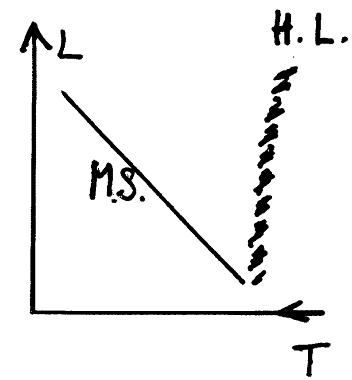
Inserting this into (*) gives

$$T_{eff} \propto M^{0.2} R^{0.06} \mu^{0.45}$$

or with all constants:

$$T_{eff} = 2.2 \times 10^3 K \left(\frac{M}{M_\odot}\right)^{0.2} \left(\frac{R}{R_\odot}\right)^{0.06} \mu^{0.45}$$

for fully convective stars. This corresponds to a nearly vertical line in the Hertzsprung–Russel diagram, it is called the *Hayashi*–line. The position of the line depends only slightly on the stellar parameters and is located around $T_{eff} \approx 2000K$.



Note: *To the right of the Hayashi–line there is no hydrostatic equlibrium configuration for stars.*

Reason: Consider three stars with the same L and M: one on the Hayashi line, the others slightly left and right aside. The star on the left has higher T_{eff} and is partly radiative because $\frac{d\ln T}{d\ln P}\big|_{rad} < 1 - \frac{1}{\gamma}$. Integrated over the star we have $\left\langle \frac{d\ln T}{d\ln P} \right\rangle_r$ left of the Hayashi line smaller as on the Hayashi line for the fully convective star. Because of continuity reasons on the right side of the Hayashi line $\left\langle \frac{d\ln T}{d\ln P} \right\rangle_r$ is larger than on the line. This means that somewhere in the star it should apply: $\frac{d\ln T}{d\ln P}\big|_{rad} > 1 - \frac{1}{\gamma}$ This is impossible because (a) if the star is fully convective, this would force $\frac{d\ln T}{d\ln P} > 1 - \frac{1}{\gamma}$, or (b) the star would get immediately fully convective which would move it onto the Hayashi line.

→ stars to the right of the Hayashi line are not stable.