

I*M*P*R*S on ASTROPHYSICS at LMU Munich

Astrophysics Introductory Course

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Chapter 1

Matter and Radiation

For a comprehensive treatment of this subject see:

Rybicki and Lightman: Radiative Processes in Astrophysics, Wiley, New York 1979

1.1 Kinetic theory of free particles

To understand astrophysical plasmas, and especially stars, we need to know their equations of state, i.e. the relations between density ρ , temperature T , pressure P and energy density u :

$$\left. \begin{array}{l} \text{Pressure } P = P(\rho, T) \\ \text{Energy density } u = u(\rho, T) \end{array} \right\} \text{Equations of State}$$

For the classical ideal gas we have (k = Boltzmann's constant):

$$\left. \begin{array}{l} P_{gas} = nkT \\ u_{gas} = \frac{3}{2}nkT = \frac{3}{2}P \end{array} \right\} \text{Classical Ideal Gas}$$

The classical ideal gas law applies for most hydrogen burning stars (main sequence stars, see below).

Generally, an equation of state for a gas can be derived with kinetic theory from the Momentum or energy distribution function of the particles.

Consider a cube of volume L^3 with N homogeneously distributed particles, i.e. we have the particle density $n_0 = N/L^3$. Provided the distribution function of the momenta $n(p)$ is

isotropic we can calculate:

$$n_o = \int_{-\infty}^{\infty} n(p) d^3 p = \int_0^{\infty} n(p) 4\pi p^2 dp$$

The pressure P on a wall is determined by the transferred momentum $d\tilde{p}$ per time interval dt and per area L^2 :

$$P = \frac{F}{L^2} = \frac{1}{L^2} \frac{d\tilde{p}}{dt}$$

The momentum transferred to the wall perpendicular to the x-direction is:

$$d\tilde{p} = \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{2p_x}_{\text{Term 1}} \underbrace{v_x dt L^2}_{\text{Term 2}} \underbrace{n(p) dp_x dp_y dp_z}_{\text{Term 3}}$$

which is readily understood if we consider that:

Term 1 = transferred momentum per particle

Term 2 = all particles in this volume reach the wall during dt

Term 3 = density of particles with momentum p

For an isotropic distribution of momenta, we can write in spherical polar coordinates:

$$p_x = p \sin \theta \cos \varphi, \quad v_x = v(p) \sin \theta \cos \varphi, \quad d^3 p = p^2 \sin \theta d\theta d\varphi dp$$

and obtain for the pressure:

$$P = \frac{1}{L^2} \frac{d\tilde{p}}{dt} = \int_0^\infty \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} 2pv p^2 n(p) \sin^3 \theta \cos^2 \varphi d\theta d\varphi dp$$

i.e.

$$P = \frac{1}{3} \int_0^\infty p \cdot v(p) n(p) 4\pi p^2 dp$$

The pressure is determined by the momentum distribution function of the particles.

The **energy density** of the particles is:

$$u_{kin} = \int_0^\infty \varepsilon(p) n(p) 4\pi p^2 dp$$

where $\varepsilon(p)$ = kinetic energy of particle with momentum p . In the non-relativistic case we have of course $\varepsilon = p^2/2m$. For relativistic particles we need:

$$\varepsilon(p) = p \cdot c \left(\sqrt{1 + \left(\frac{m_0 c}{p} \right)^2} - \frac{m_0 c}{p} \right), \quad p \cdot v = \frac{p \cdot c}{\sqrt{1 + \left(\frac{m_0 c}{p} \right)^2}}$$

1.2 Equilibrium distribution functions of fermions and bosons

All particles known obey either Bose-Einstein or Fermi-Dirac statistics. At low temperatures, the nature of the particles is important for determining their thermodynamic behaviour. At high temperatures, all ideal gases of free particles behave in the same way, i.e. like the classical ideal gas.

Using the grand canonical partition function one can show that fermions and bosons have the following energy distribution functions in thermodynamical equilibrium:

$$dN = \frac{dg}{e^{-\eta+E/kT} \pm 1}$$

E	= energy of the particle
dN	= number of particles p in energy range (E,E + dE)
dg	= number of quantum states in energy range (E,E + dE) = $\alpha d^3x d^3p/h^3$ (multiplicity α due to particle spin)
+1 in denominator	= Fermions, Pauli-principle, only one particle per phase space cell h^3
-1 in denominator	= Bosons, no Pauli-principle
η	= "Degeneracy parameter" = (chemical potential μ)/(kT)

For all massive particles which are neither created nor destroyed, it is determined from the requirement of particle number conservation:

$$N = \int \frac{dg}{e^{-\eta+E/kT} \pm 1} = \text{const.}$$

This does not apply for photons in a black body environment which have $\eta = 0$.

Depending on η we can identify:

$\eta \gg 1$	highly degenerate systems
$-5 < \eta < 5$	medium to weakly degenerate systems
$\eta \ll -1$	non-degenerate systems

(see below for explanation).

1.2.1 Momentum distributions of non-degenerate free particles

If $\eta \ll -1$, we have $\exp(-\eta + E/kT) \gg 1$ and consequently:

$$dN \propto e^{\eta} e^{-E/kT}$$

As the kinetic energy for free particles is:

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{p_1^2 + p_2^2 + p_3^2}{2m}$$

and the number of quantum states within $(E, E + dE)$ is:

$$dg = \alpha \frac{d^3x d^3p}{h^3}$$

we have:

$$dN = \alpha \frac{d^3x d^3p}{h^3}$$

Integrating yields:

$$N = \frac{\alpha}{h^3} e^{\eta} V (2\pi mkT)^{3/2}$$

which we can use to eliminate e^{η} and to obtain the well-known **Maxwell distribution**:

$$dN = \frac{N}{V} \cdot \frac{1}{(2\pi mkT)^{3/2}} \exp\left(-\frac{p^2}{2mkT}\right) d^3p dV$$

This is the energy distribution function of non-degenerate free particles in thermodynamic equilibrium.

1.2.2 Criteria for the degeneracy of a free particle gas

We first define the following two parameters to simplify the discussion:

Typical de Broglie wavelength $\lambda_{deBroglie}$ of a particle in a thermodynamical plasma:

$$\lambda_{deBroglie} = \frac{h}{\sqrt{2\pi mkT}}$$

Mean particle separation r_o :

$$n = N/V = 1/r_o^3$$

In the previous subsection we have shown that for large negative η :

$$\eta \ll -1 : \quad e^\eta = \frac{n}{\alpha} \frac{h^3}{(2\pi mkT)^{3/2}}$$

which can be rewritten as:

$$\eta = \ln \left(\frac{1}{\alpha} \frac{\lambda_{deBroglie}^3}{r_o^3} \right)$$

Therefore, $\eta \ll -1$ implies a large separation of the particles, in which case their quantum nature is not anymore relevant and we can treat them as classical particles.

Degenerate systems in astrophysics are mostly **fermionic**. Then the Pauli principle applies and the maximum phase space density is α/h^3 , i.e. one particle per phase space cell.

For fermions we have for the energy ($\varepsilon = p^2/2m$) distribution function:

$$f(\varepsilon) = \left[\exp\left(-\eta + \frac{\varepsilon}{kT}\right) + 1 \right]^{-1} = \left[\exp\left(\frac{\varepsilon - \mu}{kT}\right) + 1 \right]^{-1}$$

where we have used $\eta = \mu/(kT)$.

Complete degeneracy is approached if the system is cooled to temperatures which are much smaller than the chemical potential μ , i.e. $\eta \gg 0$ (see also figure below):

$$kT \ll \mu \quad (\eta \gg 0) : \quad f(\varepsilon) \rightarrow \begin{cases} 1 & \text{if } \varepsilon < \mu \\ 0 & \text{if } \varepsilon > \mu \end{cases}$$

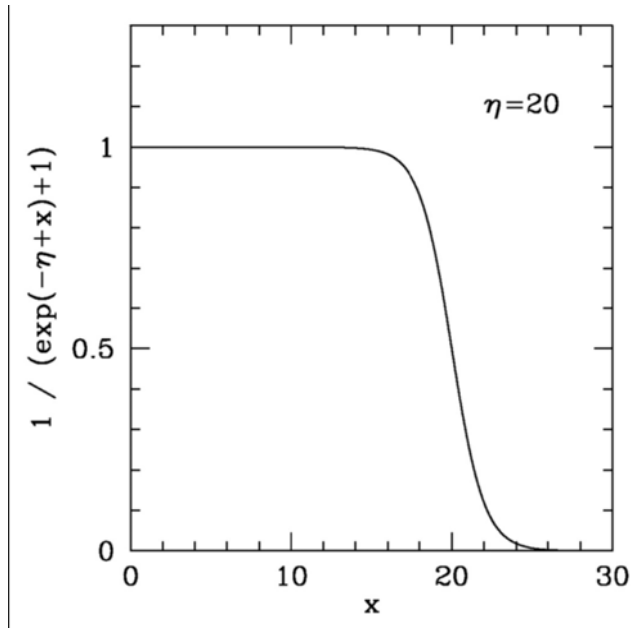
In this case, μ is called the **Fermi energy** $\varepsilon_F = \mu$, the corresponding momentum is called the **Fermi momentum** p_F . ε_F is the energy of the most energetic particle in the system (remember: $dU = TdS - PdV + \mu dN$).

We can determine η by requiring that the integral over the occupation numbers is N :

$$N = \frac{\alpha}{h^3} \iint \frac{1}{\exp\left(-\eta + p^2 / 2mkT\right) + 1} d^3 p d^3 x$$

If the distribution of momenta is isotropic and the particle distribution homogeneous, this yields:

$$n = \frac{N}{V} = \frac{\alpha}{h^3} \int \frac{1}{\exp(-\eta + p^2 / 2mkT) + 1} 4\pi p^2 dp$$



Substituting $x = p^2/(2mkT)$ we obtain:

$$n = \frac{\alpha}{h^3} \frac{2}{\pi^{1/2}} (2\pi mkT)^{3/2} \int \frac{x^{1/2} dx}{\exp(-\eta + x) + 1}$$

For strong degeneracy $\eta \gg 1$, $1/(\exp(-\eta + x) + 1)$ can be approximated by a step function (see figure) and the integral simplifies to:

$$\int \frac{x^{1/2} dx}{\exp(-\eta + x) + 1} = \int_0^\eta x^{1/2} dx = \frac{2}{3} \eta^{3/2}$$

Solving for η and inserting $\lambda_{\text{deBroglie}}$ and r_o we finally have:

$$\eta = \frac{1.21}{\alpha^{2/3}} \left(\frac{\lambda_{\text{deBroglie}}}{r_o} \right)^2$$

Therefore, $\eta \gg 1$, if $r_o \ll \lambda_{\text{deBroglie}}$. The quantum mechanical nature of particles cannot be neglected anymore if their separation is smaller than their typical de Broglie wavelength.

We can calculate the **Fermi momentum** p_F from the fully degenerate case:

$$n = \frac{\alpha}{h^3} \int_0^{p_F} n(p) 4\pi p^2 dp = \frac{\alpha}{h^3} \frac{4\pi}{3} p_F^3$$

or:

$$p_F = h \left(\frac{3}{\alpha 4\pi} \right)^{1/3} n^{1/3}$$

The Fermi momentum is the highest particle momentum in case of infinite degeneracy.

Bottom line: $(\lambda_{\text{deBroglie}}/r_o)$ **determines the degree of degeneracy:**

$$\frac{\lambda_{\text{deBroglie}}}{r_o} \begin{cases} \gg 1 & \text{Fermi gas} \\ \ll 1 & \text{Maxwell/Boltzmann gas} \end{cases}$$

1.2.3 Astrophysical examples of degenerate fermions

We consider a white dwarf, a neutron star and the sun, all at a mass of one solar mass. The mean densities and temperatures are

$$n = \frac{1}{m_p} \frac{M_\odot}{\frac{4\pi}{3} R^3}, \quad \bar{T} \approx 2 \cdot 10^7 K$$

The ratio ($\lambda_{\text{deBroglie}}/r_o$) is:

	R	n (cm ⁻³)	$\lambda_{\text{deBroglie}}/r_o$	
			Electrons	Protons/Neutrons
Sun	$7 \cdot 10^{10}$ cm	$8 \cdot 10^{23}$	0.15	$3.7 \cdot 10^{-3}$
White Dwarf	1/100 R_\odot	$8 \cdot 10^{29}$	15	0.37
Neutron Star	$1.4 \cdot 10^{-5}$ R_\odot	$3 \cdot 10^{38}$	---	27

Degree of degeneracy

- ◆ Sun: ideal gas, Maxwell/Boltzmann
- ◆ White dwarfs: electrons degenerate, protons Maxwell
- ◆ Neutron star: neutrons degenerate

1.3 The Planck distribution function for photons

Photons are bosons. Because their particle number is not conserved in thermodynamic equilibrium, we have $\eta = 0$ and, so, their distribution function is:

$$dN = \frac{dg}{e^{E/kT} - 1}$$

Considering that in full thermodynamical equilibrium the distribution of momenta is isotropic and that photons have two directions of polarization, we obtain for dg :

$$dg = 2dV \frac{d^3 p}{h^3} = \frac{2dV}{h^3} 4\pi p^2 dp$$

Energies, momenta and frequencies of photons are related via:

$$E = h\nu, \quad p = \frac{h\nu}{c}, \quad dp = \frac{h}{c} d\nu$$

Inserting yields the **number density of photons in the frequency interval** ($\nu, \nu+d\nu$):

$$dn = \frac{dN}{dV} = \frac{4\pi}{c} \frac{2\nu^2}{c^2} \frac{1}{e^{h\nu/kT} - 1} d\nu$$

Multiplication with h results in the **energy density of photons in the frequency interval $(\nu, \nu+d\nu)$** :

$$U_\nu d\nu = \frac{4\pi}{c} \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} d\nu$$

If we define the **intensity** $B_\nu(T)$ as the energy which flows through a unit surface per second and per solid angle, we obtain for the relation between energy density and intensity (without derivation):

$$U_\nu = \frac{1}{c} \int_{4\pi} B_\nu d\omega = \frac{4\pi}{c} B_\nu$$

$B_\nu(T)$ is the **Planck function of photons in thermodynamic equilibrium (black body radiation)**:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1}$$

Note: the frequency distribution of radiation of a system in thermodynamic equilibrium is

- isotropic
- homogeneous
- independent of chemical composition of emitting material
- only dependent on temperature

For high and low frequencies we have:

◆ **Rayleigh-Jeans approximation**

$$\frac{h\nu}{kT} \ll 1 \Rightarrow e^{h\nu/kT} \approx 1 + \frac{h\nu}{kT}$$

$$\Rightarrow B_\nu(T) = \frac{2\nu^2}{c^2} kT$$

◆ **Wien's approximation**

$$\frac{h\nu}{kT} \gg 1 \Rightarrow e^{h\nu/kT} \gg 1$$

$$\Rightarrow B_\nu(T) = \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

Integration over frequency yields the **Stefan-Boltzmann law for black body radiation**:

$$B(T) = \int_0^{\infty} B_\nu(T) d\nu = \sigma_B T^4$$

1.4 Equations of state

- For the Maxwell gas we have:

$$n(p) = \frac{1}{(2\pi mkT)^{3/2}} e^{-\frac{p^2}{2mkT}}$$

Using $pv = p^2/m$, $\varepsilon = p^2/(2m)$ and the pressure equation from kinetic theory:

$$P = \frac{1}{3} \int_0^{\infty} p \cdot v(p) n(p) 4\pi p^2 dp$$

we obtain:

$$P = \frac{1}{3} \int_0^{\infty} pvn(p) 4\pi p^2 dp = nkT$$

and

$$U_{kin} = \int_0^{\infty} \varepsilon(p) n(p) 4\pi p^2 dp = \frac{3}{2} nkT = \frac{3}{2} P$$

- For a **non-relativistic degenerate Fermi gas** we obtain analogously:

$$P = \frac{1}{3} \frac{2}{h^3} \frac{4\pi}{m} \int_0^{p_F} p^4 dp = \frac{1}{3} \frac{2}{h^3} \frac{4\pi}{m} \frac{1}{5} p_F^5$$

(where p_F is the Fermi momentum as defined above), or:

$$P = \frac{8\pi}{15} h^2 \left(\frac{3}{8\pi} \right)^{5/3} \frac{1}{m} n^{5/3} = \frac{2}{3} U_{kin}$$

(calculation of U_{kin} not shown).

- For very high densities, we get a **relativistic degenerate Fermi gas**, i.e. $p_F \gg m_0 c$. We then have to use the relativistic $\varepsilon(p)$: $p v(p) = pc(1 - m_0^2 c^2 / 2p^2)$ (Taylor series of square root) to get:

Yielding

$$P = \frac{1}{3} \frac{2}{h^3} 4\pi c \int_0^{p_F} p^3 \left(1 - \frac{m_0^2 c^2}{2p^2} \right) dp$$

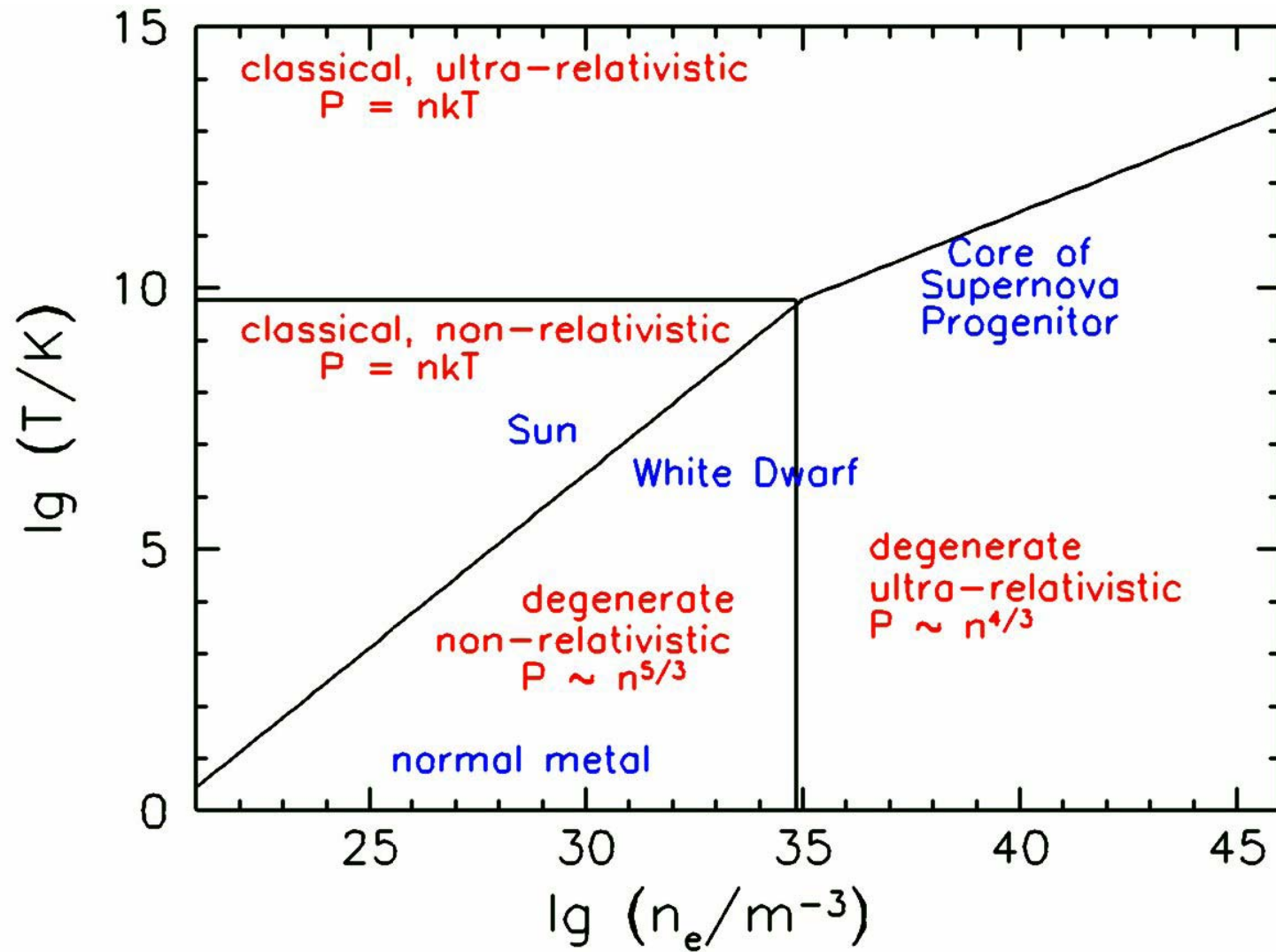
$$P = \frac{2}{3} \frac{\pi}{h^3} c p_F^4 \left(1 - \frac{m_0^2 c^2}{p^2} \right) = \frac{1}{3} U_{kin} \left(1 + \frac{4m_0 c}{3p_F} \right)$$

Or, as a function of n :

$$P = \frac{2}{3} c h \left(\frac{3}{8\pi} \right)^{4/3} n^{4/3} \left(1 - n^{-2/3} \frac{m_0^2 c^2}{h^2 \left(\frac{3}{8\pi} \right)^{2/3}} \right)$$

and

$$P = \frac{1}{3} U_{kin} \left(1 + n^{-1/3} \frac{4m_0 c}{3h \left(\frac{3}{8\pi} \right)^{1/3}} \right)$$



Temperature density diagram, adapted from Phillips (see Bibliography)

Summary: equations of state

- Normal stars: $P = nkT$, $\rho = \mu m_p n$, $U_{\text{kin}} = 3/2 \cdot P$. (μ is the mean particle weight; it is operationally defined via $\mu = \rho/(m_p n)$)
- White dwarfs: Degenerate electron gas supplies pressure, $n = n_e$, $\rho = \mu_e m_p n_e$, $\mu_e = z\mu$, where z is the number of free electrons per atom.
 - non-relativistic degeneracy ($\rho < 10^6 \text{gcm}^{-3}$): $P = A_1 \rho^{5/3}$, $U_{\text{kin}} = 3/2 \cdot P$
 - relativistic degeneracy ($\rho > 10^6 \text{gcm}^{-3}$): $P = A_2 (\rho^{4/3} - B_2 \rho^{2/3})$, $P = 1/3 \cdot U_{\text{kin}} (1 + C_2 \rho^{-1/3})$
- Neutron star: Degenerate neutrons supply pressure, $n = n_n$, $\rho = n m_n$, formulae like white dwarf, but with different coefficients A_1, A_2, B_2 and critical density.
- Photons: photons are bosons, therefore

- $$n(p) = \frac{2}{h^3} \frac{1}{e^{hv/kT} - 1}$$

- $$U_{\text{phot}} = \frac{4\pi}{c} \int B_\nu(T) d\nu \propto T^4$$

- $$P_{\text{phot}} = \frac{1}{3} U_{\text{phot}}$$

1.5 The Boltzmann formula

Consider free non-degenerate atoms with an excited electron. The energy of one atom A is the sum of its kinetic energy plus the excitation energy E_i :

$$E = \frac{p_A^2}{2m_A} + E_i$$

The number of quantum states within $(E, E + dE)$ is:

$$dg_A = \frac{d^3x d^3p_A}{h^3} g_i$$

where g_i is the degeneracy of the excited level (see below). Integrating over the distribution function gives:

$$\begin{aligned} N_i &= \int dN_i = \frac{g_i}{h^3} e^{n_A} e^{-E_i/kT} \iint e^{-p_A^2/(2m_A kT)} d^3p_A d^3x \\ &= V \frac{(2\pi m_A kT)^{3/2}}{h^3} e^{n_A} g_i e^{-E_i/kT} \end{aligned}$$

or:

$$n_i = \frac{N_i}{V} = \frac{(2\pi m_A kT)^{3/2}}{h^3} e^{n_A} g_i e^{-E_i/kT}$$

where n_i gives the number density of atoms with an electron in quantum state i . The number density ratio of atoms in two different energy levels E_i and E_j is:

$$\boxed{\frac{n_j}{n_i} = \frac{g_j}{g_i} \exp\left(-\frac{E_j - E_i}{kT}\right)}$$

which is the famous **Boltzmann-formula for the ratio of occupation numbers**.

To obtain the number density $n_{A,i}$ of an atom or ion A in a quantum state i relative to the number density n_A of all atoms or ions A , we first have to calculate the sum over all quantum states, *i.e.* to derive the **partition function**:

$$Z_A(T) = \sum_{i=0}^{\infty} g_{A,i} \exp\left(-\frac{E_{A,i}}{kT}\right)$$

With Z_A we then have:

$$\frac{n_{A,i}}{n_A} = \frac{g_{A,i}}{Z_A(T)} \exp\left(-\frac{E_{A,i}}{kT}\right)$$

1.6 The Saha equation

We now treat the simplest case of ionisation (bound-free transition) which is described by:

atom in ground state + photon \rightarrow ionised atom in ground state + free electron

We have:

$$E_A = \frac{p_A^2}{2m_A} = \text{energy of atom}$$

$$E_I = \frac{p_I^2}{2m_A} + E_{ion} = \text{kinetic energy plus ionization energy of ion}$$

$$E_e = \frac{p_e^2}{2m_e} = \text{kinetic energy of electron}$$

where we have approximated with very good accuracy: $m_I = m_A$. The number of states is:

$$dg_A = g_A \frac{d^3 p_A d^3 x}{h^3} \quad dg_I = g_I \frac{d^3 p_I d^3 x}{h^3} \quad dg_e = g_e \frac{d^3 p_e d^3 x}{h^3}$$

As in the previous section, we integrate over the distribution function for each of these

particles to obtain the particle number densities:

$$n_A = \frac{N_A}{V} = g_A \frac{(2\pi m_A kT)^{3/2}}{h^3} e^{\eta_A}$$

$$n_I = \frac{N_I}{V} = g_I \frac{(2\pi m_A kT)^{3/2}}{h^3} e^{\eta_I} e^{-E_{ion}/kT}$$

$$n_e = \frac{N_e}{V} = 2 \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{\eta_e}$$

where we have used $g_e = 2$ for the two spin states of the electron. As energy is conserved, the chemical potentials have to fulfill the relation:

$$\eta_I + \eta_e - \eta_A = 0$$

(note that the chemical potential of photons vanishes). Therefore, the product $n_I \cdot n_e / n_A$ allows to eliminate all unknowns η and we obtain the **Saha-Equation**:

$$\boxed{\frac{n_I \cdot n_e}{n_A} = \frac{2g_I}{g_A} \frac{(2\pi m_e kT)^{3/2}}{h^3} \exp\left(-\frac{E_{ion}}{kT}\right)}$$

1.7 Radiation: basic definitions

Definition of **Intensity**:

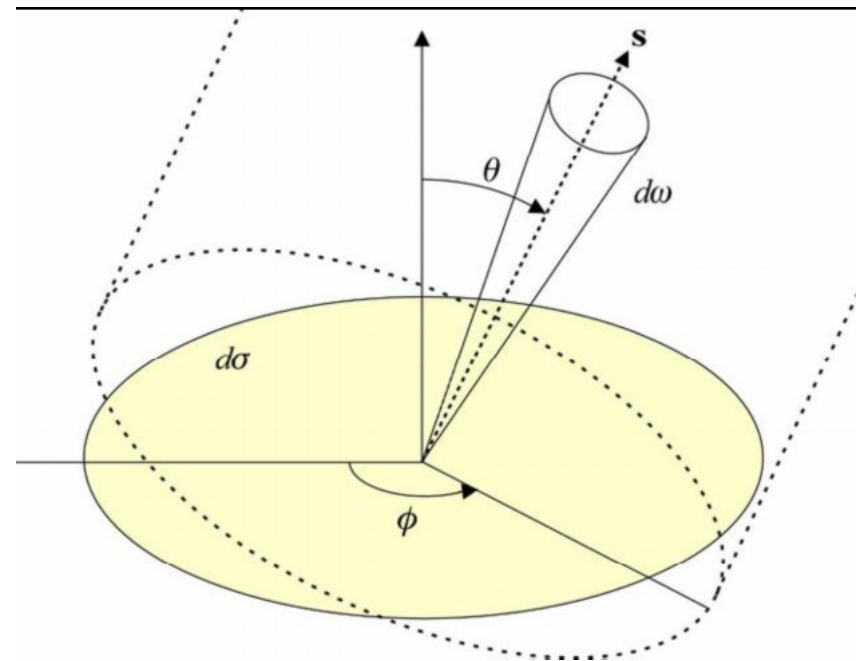
Consider energy flowing through a small surface element. We then define the **intensity** as the energy per frequency ν per time dt per solid angle $d\omega$ and per projected area $\cos\theta d\sigma$ flowing through the surface into the direction θ , i.e. via:

$$d^4 E_\nu = I_\nu \cos\theta d\sigma d\nu d\omega dt$$

I_ν	= Intensity per frequency
$d\sigma$	= Surface of receiver at r
$d\omega$	= Solid angle within which radiation is received
θ, ϕ	= Angles with respect to the norm of $d\sigma$
ν	= frequency
t	= Time

The intensity has the following properties:

- $\dim [I_\nu] = \text{erg cm}^{-2} \text{ s}^{-1} \text{ Hz}^{-1} \text{ sterad}^{-1}$



The solid angles under which the area elements appear are:

$$d\omega = r^{-2} \cos \theta' d\sigma'$$

$$d\omega' = r^{-2} \cos \theta d\sigma$$

(r = distance between the surfaces) which we can insert above and obtain:

$$I'_\nu = I_\nu$$

This is nothing else than the **conservation of surface brightness!**

● **Radiation flux density** of a star:

The energy flux through $d\sigma$ in all directions is:

$$d^3 E_\nu = \int_{4\pi} d^4 E_\nu d\omega$$

which with the definition of I_ν yields:

$$d^3 E_\nu = d\nu dt d\sigma \int_{4\pi} I_\nu \cos \theta d\omega$$

One now defines the radiation flux density F_ν as

$$\mathbf{F}_\nu = \int_{4\pi} I_\nu \cos \theta d\omega$$

This is an integral over the full sphere, i.e. it represents the **net** energy emitted through the area $d\sigma$ (energy emitted minus energy received). For stars, the received energy can be neglected.

- **Luminosity** of a star:

We obtain the luminosity per frequency of the star via multiplication with the whole surface area (R = radius of the star):

$$L_\nu = 4\pi R^2 F_\nu$$

Further integration over ν gives the total luminosity:

$$L = 4\pi R^2 \int_\nu F_\nu d\nu$$

- **Effective temperature** of a star:

If a star emits approximately black body radiation, we can write for the total luminosity according to the Stefan-Boltzmann-law:

$$L = 4\pi R^2 \sigma_B T_{eff}^4$$

This is the defining equation for the **effective temperature** T_{eff} . This equation implies:

$$\sigma_B T_{eff}^4 = \int_\nu F_\nu d\nu = \int_\nu \int_{4\pi} I_\nu \cos \theta d\omega d\nu$$

and

$$F_{\nu} = B_{\nu} = \int_{4\pi} I_{\nu} \cos \theta d\omega$$

where B_{ν} is the Planck radiation law (see below) and $\sigma_B = 5.67 \cdot 10^{-5} \text{ erg/s/cm}^2/\text{K}^4$ is the Stefan-Boltzmann constant.

● Flux of a star received on earth:

The flux per frequency received on earth is the luminosity emitted per frequency divided by the surface area of a sphere with a radius corresponding to the distance r between star and earth:

$$f_{\nu} = \frac{L_{\nu}}{4\pi r^2} = \frac{R^2}{r^2} F_{\nu} = \frac{R^2}{r^2} B_{\nu}$$

where the latter equation only holds if the star emits black body radiation.

1.8 Atomic and Molecular Transitions

Emission and absorption of photons occurs through processes in atoms, molecules, and solid bodies. These processes are governed by Quantum Mechanics. We have a:

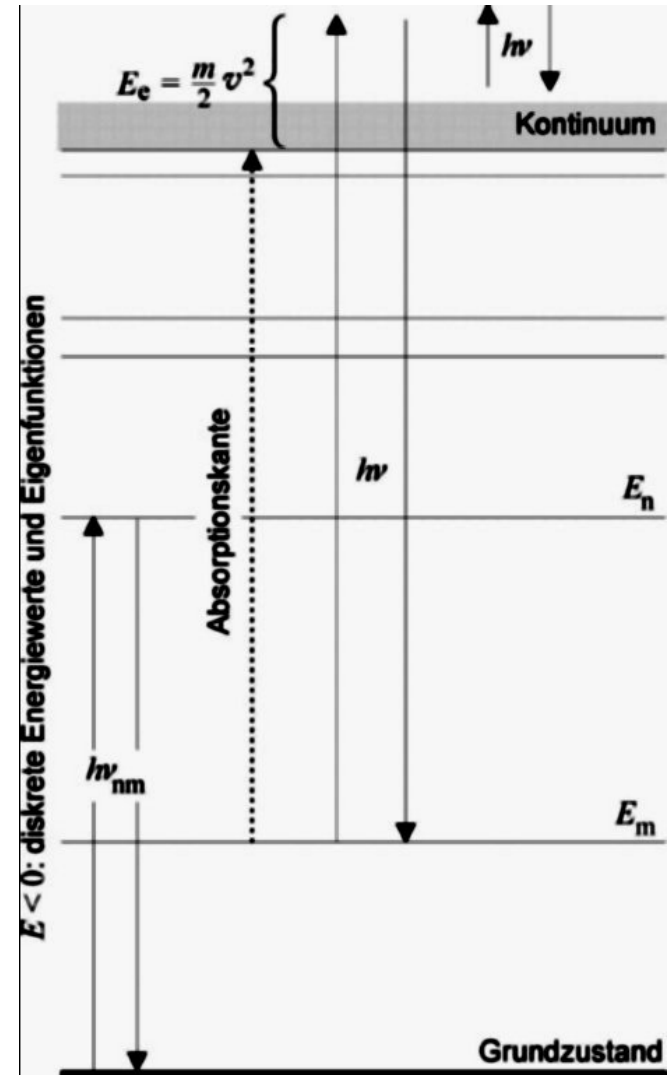
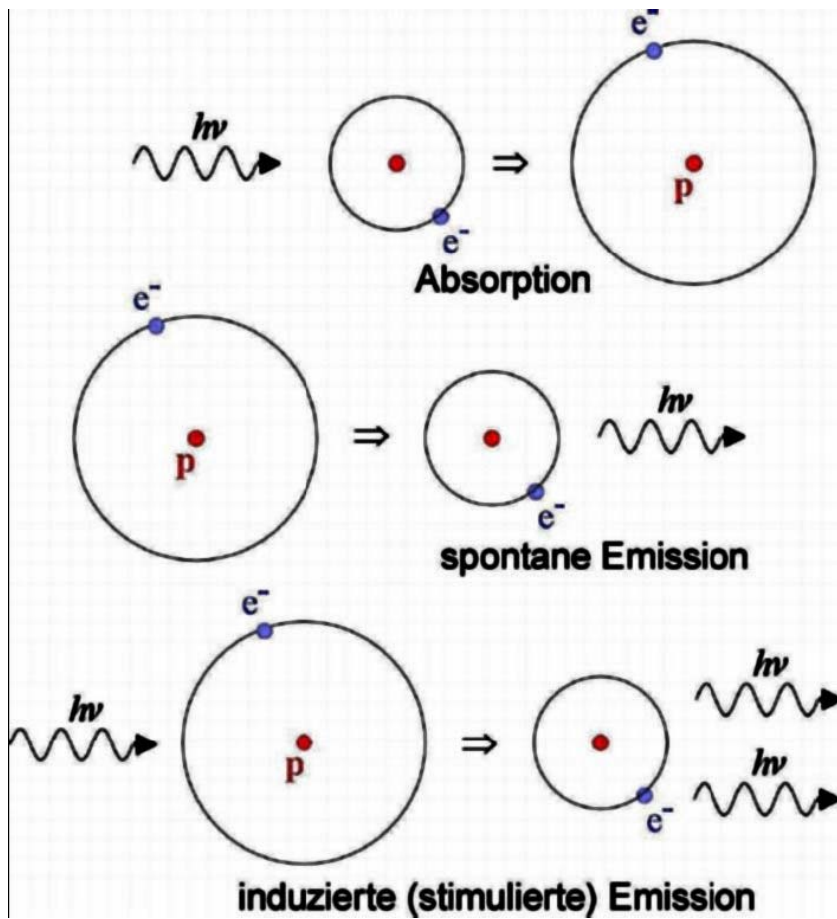
- **discrete** spectrum of energy eigenvalues for bound electrons ($E < 0$)
- **continuous** energy spectrum for free electrons ($E > 0$)

The following interactions between photons and electrons are possible (provided the quantum mechanical transition rules are fulfilled):

- **absorption**
- **spontaneous emission**
- **stimulated emission**

whereby these **transitions** can occur between the following energy levels:

- **discrete-discrete (bound-bound) → spectral lines**
- **discrete-continuous (bound-free) → ionization/recombination continua**
- **continuous-continuous (free-free) → “Bremsstrahlung” (ion required!)**



1.8.1 Eigenvalues of hydrogen and H-like ions (only one e^-)

- The state of the electron is described by the following **quantum numbers**:

n	Main quantum number	$n = 1, 2, 3, \dots$
l	Orbital angular momentum	$l = 0, 1, 2, \dots, n - 1$
m_l	z-component of orbital angular momentum	$m_l = -l, -(l - 1), \dots, l - 1, l$
s	Electron spin	$s = \pm 1/2$

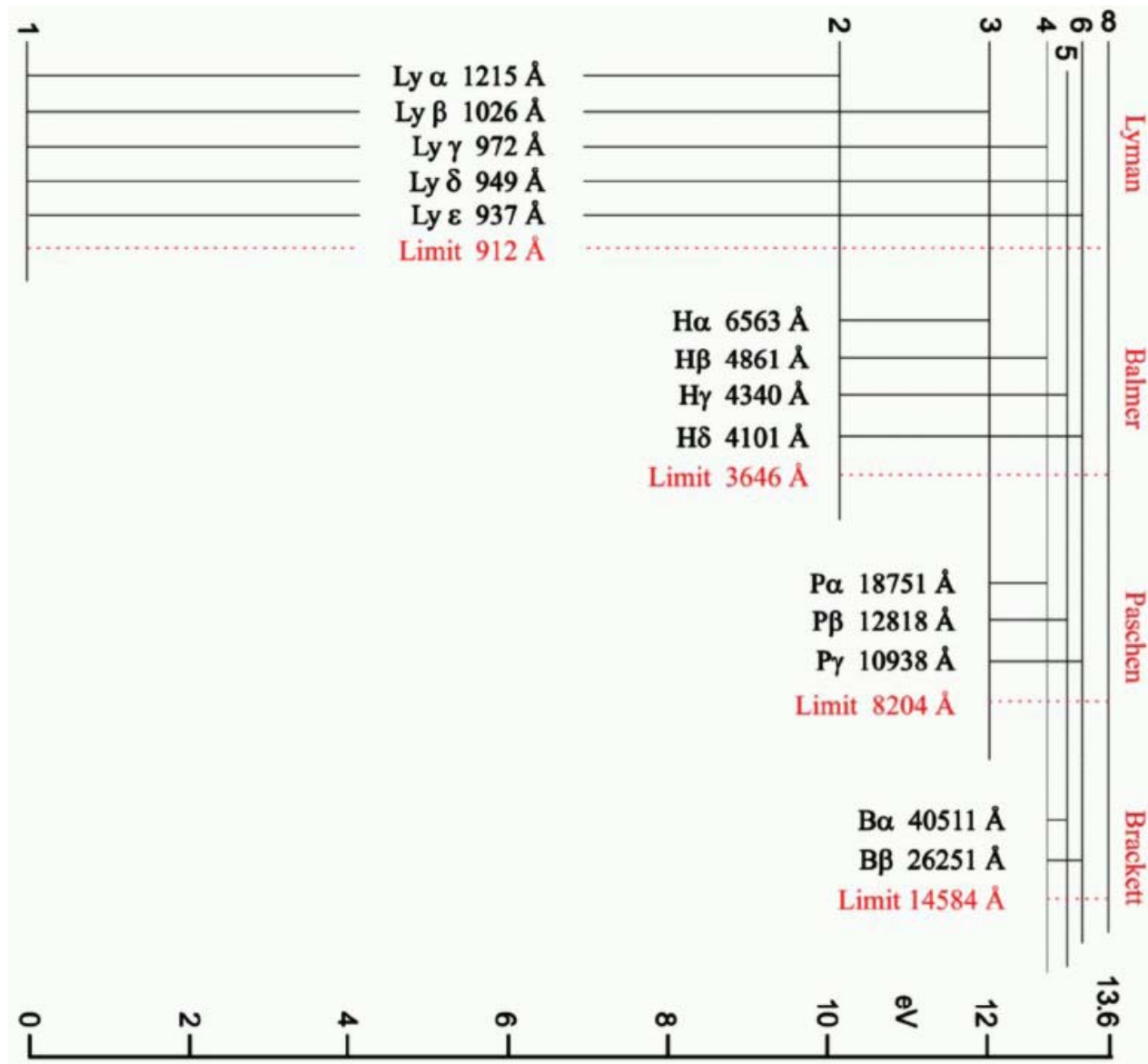
- Energy Eigenvalues ($a_0 = \text{Bohr radius} = 0.529 \text{ \AA}$)

$$E_n = -\frac{e^2 Z^2}{2a_0 n^2}$$

- Balmer lines: $n = 2 \rightarrow n = 3, 4, 5, \dots$ ($H\alpha, H\beta, H\gamma, \dots$)

- Degree of degeneracy per energy eigenvalue

$$g_n = 2 \sum_{l=0}^{n-1} (2l + 1) = 2n^2$$



1.8.2 Eigenvalues of atoms with more electrons

- N(electrons) = 1
 - Alkali metals
 - Electrostatic shielding of nuclear potential

- N(electrons) > 1
 - LS-coupling (spin-orbit « Coulomb)

$$\vec{L} = \sum \vec{l}_i, \quad \vec{S} = \sum \vec{s}_i, \quad \vec{J} = \vec{L} + \vec{S}$$

- jj-coupling (spin-orbit » Coulomb)

$$\vec{j}_i = \vec{l}_i + \vec{s}_i, \quad \vec{J} = \sum \vec{j}_i$$

1.8.3 Eigenvalues of molecules

- Molecular transitions important in interstellar medium and cool stars
- transition through vibration (near IR) and rotation (submm, mm, radio)
- $E_{\text{el}} \gg E_{\text{vib}} \gg E_{\text{rot}}$

1.8.4 Probability of line transitions

The probability for a line transition $l \rightarrow u$ ($u \rightarrow l$) through absorption (emission) of a photon with frequency (ν , $\nu+d\nu$) from (into) the solid angle (ω , $\omega+d\omega$) is:

$$d^2W_{ul}^{sp} = A_{ul} \varphi(\nu) d\nu \frac{d\omega}{4\pi} \quad (\text{spontaneous emission})$$

$$d^2W_{ul}^{stim} = B_{ul} I_\nu \varphi(\nu) d\nu \frac{d\omega}{4\pi} \quad (\text{stimulated emission})$$

$$d^2W_{lu}^{abs} = B_{lu} I_\nu \varphi(\nu) d\nu \frac{d\omega}{4\pi} \quad (\text{absorption})$$

where A_{ul} , B_{ul} , B_{lu} are the **Einstein Coefficients** which are related via:

$$A_{ul} = \frac{2h\nu^3}{c^2} \frac{g_l}{g_u} B_{lu}$$

$$g_u B_{ul} = g_l B_{lu}$$

with: h = Planck's constant
 g_l, g_u = statistical weights of the levels l, u = degree of degeneracy

A_{ul} needs to be derived with quantum mechanical methods.

$\phi(\nu)d\nu$ describes the probability for absorption or emission with frequencies $(\nu, \nu+d\nu)$. $\phi(\nu)$ is needed because emission and absorption lines are broadened. $\phi(\nu)$ is normalized via:

$$\int_{-\infty}^{\infty} \phi(\nu) d\nu = 1$$

1.8.5 Line broadening

Lines are broadened because of:

- Heisenberg's uncertainty principle: the excited level has finite lifetime $\tau_{\text{natural}} = 1/A_{ul}$ which implies an energy uncertainty (**natural line width**) ΔE of:

$$\Delta E \cdot \tau \geq h \quad (\text{Heisenberg's Uncertainty Relation})$$

Using quantum mechanics we can derive the line profile which has the following shape:

$$\phi(\nu) = 2\pi^{-1} \frac{A_{ul} / 2}{(\nu - \nu_c)^2 + (A_{ul} / 2)^2}$$

This is called a **damping or Lorentzian profile**. In classical damping theory, an oscil-

lating and radiating electron would have:

$$A_{ul} = \frac{8\pi^2 e^2}{3m_e c \lambda_0^2}$$

where λ_0 is the central wavelength of the line.

- Thermal motion of atoms imply a Maxwell distribution of their velocities (in thermal equilibrium):

$$\Pi(v_x, v_y, v_z) = \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{m}{2kT}(v_x^2 + v_y^2 + v_z^2)}$$

The Doppler-effect then causes a frequency distribution:

$$\varphi(\nu) = \pi^{-1/2} \frac{1}{\Delta\nu_{therm}} e^{-\left(\frac{\nu - \nu_c}{\Delta\nu_{therm}}\right)^2}$$

with a **thermal line width** of:

$$\frac{\Delta\nu_{therm}}{\nu} = \sqrt{\frac{2kT}{m_{Atom} c^2}}$$

- Perturbations of the energy levels through electromagnetic interaction with neighbouring particles (atoms, ions, electrons), Stark-effect. We have:

$$\varphi(\nu) = 2\pi^{-1} \frac{\gamma/2}{(\nu - \nu_c)^2 + (\gamma/2)^2}$$

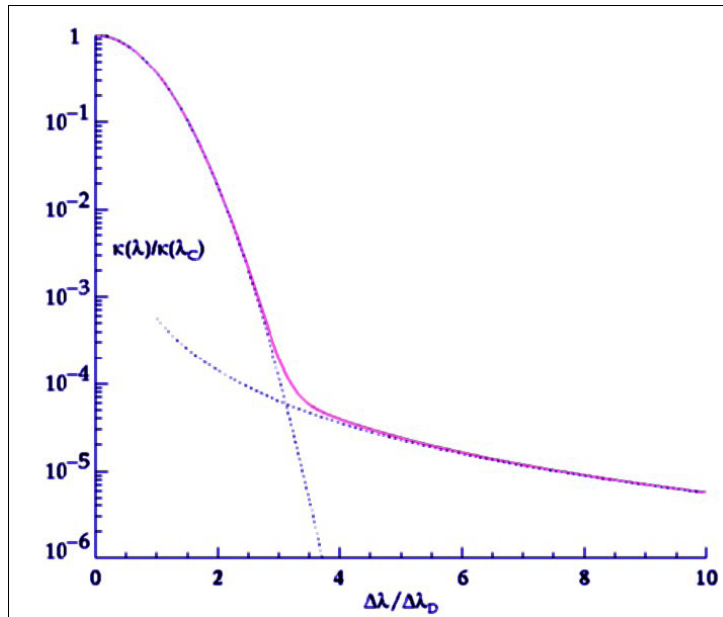
where γ is now related to the life times τ via:

$$\gamma = \frac{1}{\tau_{natural}} + \frac{1}{\tau_{pressure}}$$

and the **pressure induced line width** is:

$$\tau_{pressure} \simeq \frac{1}{\text{particle density}}$$

- The final broadening function $\phi(\nu)$ is obtained via convolution of this latter broadening function with the thermal broadening function → **Voigt profile**:



$$\phi(\nu) = \frac{1}{\sqrt{\pi} \Delta \nu_D} \frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-y)^2 dy}{\alpha^2 + (x - y)^2}$$

$$\alpha = \frac{1}{4\pi} \frac{\gamma}{\Delta \nu_D}$$

$$x = \frac{\nu - \nu_c}{\Delta \nu_D}$$

- Width of line core proportional to $T^{1/2}$ or particle velocity.
- Width of wings proportional to particle density.

1.8.6 Interaction cross sections

- **Interaction cross sections for line transitions.** The reaction probability dW_{lu} for an atom to be excited by a parallel beam of radiation can be written as:

$$dW_{lu}(\nu) = \sigma_{lu}(\nu)cn_{\nu}(\nu)d\nu$$

where $\sigma_{lu}(\nu)$ is the cross section for the reaction and $cn_{\nu}(\nu)d\nu$ is the photon flux in $(\nu, \nu+d\nu)$, i.e. the number density of photons in $(\nu, \nu+d\nu)$ times the speed of light. $cn_{\nu}d\nu$ we can rewrite as

$$cn_{\nu}d\nu = c \frac{dE / (h\nu)}{cdtdA} = \frac{dE}{h\nu dtdA}$$

where $cdtdA$ is the volume element we consider and of which dA is chosen to be the surface element *perpendicular* to the direction of radiation. Using the definition of the intensity (for a parallel beam there is no $d\omega$) and taking into account $dA = d\sigma \cos \theta$ we obtain:

$$cn_{\nu} = \frac{I_{\nu}}{h\nu}$$

Comparing this with the initially given definition of dW_{lu} , we finally arrive at:

$$\sigma_{lu} = \frac{h\nu}{4\pi} B_{lu} \varphi(\nu)$$

- **Interaction cross sections for bound-free transitions.** Be ν_l the frequency corresponding to the ionization energy. Then, in case of hydrogen, we can use the Kramers approximation for the interaction cross section:

$$\nu < \nu_l : \quad \sigma_{bf}(\nu) = 0$$

$$\nu \geq \nu_l : \quad \sigma_{bf}(\nu) = \text{const} \cdot (\nu_l / \nu)^3$$

For more complex ions higher order terms in (ν_l/ν) are needed as well. As for line transitions, there exist again simple relations between the cross sections for absorption, stimulated emission and spontaneous emission.

- **Interaction cross sections for free-free transitions (Bremsstrahlung).**

This cross section depend also on the velocities of the free electrons. We have:

$$\sigma_{bf} = \frac{\text{const}}{\nu^3} \int \frac{g_{ff}(\nu, \nu)}{\nu} n_e(\nu) d\nu$$

where n_e is the electron density per velocity interval and g_{ff} is the Gaunt-factor, a quantum-mechanical correction to the Kramers-approximation.

- **Note:** If thermodynamical equilibrium holds, we know from Wien's law that the maximum or typical frequency of the photons is proportional to the temperature $\nu \sim T$. Therefore, we expect that the opacities $\kappa \sim n\sigma$ (see below) for bound-free or free-free transitions decrease with the same power of the temperature as the power of the frequency, i.e.

$\sim T^{-3}$. Modifications from this simple expectation are introduced because, e.g., the ionization changes as a function of temperature. Therefore, the opacity for bound-free and free-free transitions actually shows $\kappa \sim \rho T^{-3.5}$.

1.9 Other Emission and Scattering Processes

1.9.1 Rayleigh Scattering

- Interaction of bound electrons with **low-energy** photons.
- Absorption or emission of a photon can be described as a resonance effect, with the interaction cross section σ being:

$$\sigma(\nu) \propto \frac{\nu^4}{(\nu^2 + \nu_0^2)^2} \quad \text{or for } \nu \ll \nu_0 : \quad \sigma(\nu) \propto \frac{\nu^4}{\nu_0^4}$$

where $h\nu_0$ corresponds to the energy of the line transition.

- Even if a photon has a frequency much lower than $\nu \ll \nu_0$, there is still a small chance of interaction. The scattering part of the interaction is called → **Rayleigh scattering**. For hydrogen, the Rayleigh cross section per atom is (now in wavelength λ):

$$\sigma_R = \frac{8\pi e^4}{3m_e^2 c^4} \frac{(1026\text{\AA})^4}{\lambda^4}$$

The reference wavelength does not just correspond to Ly α (1215Å) or the Lyman limit (912Å) because all Lyman lines contribute to the scattering.

- Rayleigh scattering is the origin of the blue sky.

1.9.2 Thomson Scattering

- Interaction of free electrons with **low-energy** photons
- This corresponds to a 'resonance' with $v_0 = 0$. Therefore we have:

$$\sigma_T = \frac{8\pi e^4}{3m_e^2 c^4} = 6.6652 \cdot 10^{-25} \text{ cm}^2$$

Note: The Thomson cross section roughly corresponds to the radius at which the electrostatic potential of the electron equals its mass.

- A better approximation for low energies is obtained from an expansion of the Klein-Nishina formula:

$$\sigma(\nu) = \sigma_T (1 - 2\alpha + (56/5)\alpha^2 + \dots) \quad \text{with} \quad \alpha = \frac{h\nu}{m_e c^2}$$

- Important for IR, optical, UV radiation in:
 - Stellar atmospheres, stellar cores
 - Interstellar and intergalactic gas
 - Quasars
 - Big Bang (coupling of photons and matter before recombination)

1.9.3 Compton Scattering

- Interaction of free electrons with **high-energy** photons

- An expansion of the Klein-Nishina formula in this case is:

$$\sigma(\nu) = \sigma_T \frac{3}{8\alpha} (\ln(2\alpha) + \frac{1}{2}) \quad \text{with} \quad \alpha = \frac{h\nu}{m_e c^2}$$

i.e.

$$0 < \sigma_c < \sigma_T \quad \forall \quad \frac{h\nu}{m_e c^2}$$

- Compton scattering is important for

- Extremely hot gas
- X-ray and γ quanta

- Interaction of **high-energy** free electrons with **low-energy** photons is called **inverse Compton effect**.

- X-ray production in jets and active galaxies
- Sunyaev-Zeldovich effect
(scattering of the 3K radiation at X-ray gas of galaxy clusters)

1.9.4 Synchrotron Radiation

- Relativistic electrons

$$E = mc^2 = m_0c^2\gamma \gg m_0c^2 = 0.51 \text{ MeV}$$

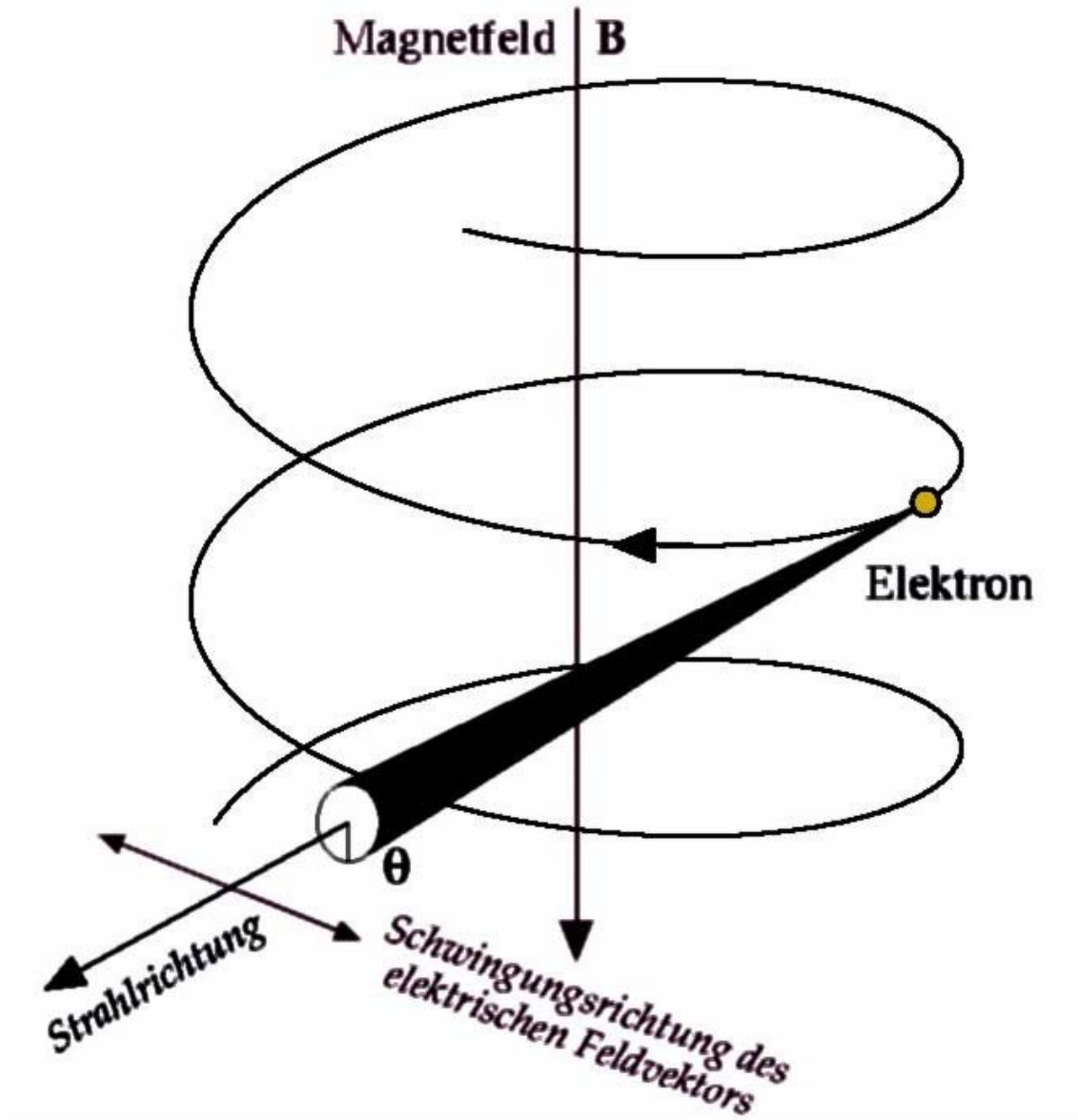
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

gyrate in a homogeneous magnetic field \mathbf{B} . As they are accelerated in this process, they emit radiation. Low-energy electrons emit cyclotron radiation, relativistic electrons emit **synchrotron radiation**.

- The radiation is emitted tangentially in a tight cone with apex angle θ

$$\theta = \frac{m_0c^2}{E} = \frac{1}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}}$$

- Because there is a preferred plane of gyration perpendicular to the magnetic field, synchrotron radiation is highly polarized. The polarization is perpendicular to the magnetic field direction.



- The observer sees a series of short flashes, implying a broad energy distribution. The peak of the spectrum is at:

$$\nu_{sync} = 0.079 \frac{eB_{\perp}}{m_e c} \gamma^2$$

- For a power-law distribution of electron energies (as e.g. observed in the cosmic ray energy distribution):

$$N(E)dE = E^{-p} dE$$

we obtain a volume emissivity (see below for exact definition) of radiation:

$$\varepsilon(\nu) \propto B_{\perp}^{(p+1)/2} \nu^{-(p-1)/2}$$

- Synchrotron radiation is observed, e.g., in
 - Neutron stars → rotating magnetosphere
 - Supernova remnants
 - Spiral galaxies → galactic magnetic fields
 - Galaxy clusters
 - Active galactic nuclei, jets
- Synchrotron radiation is a typical form of non-thermal radiation.

1.10 Radiative Transfer

1.10.1 Absorption coefficient

The intensity I_ν changes, when a light beam passes through matter. Photons can be absorbed, emitted and scattered.

We expect that the reduction in intensity dI_ν will be proportional to the intensity I_ν itself and proportional to the path length ds through the material (this may not be correct for very high intensities!). Therefore, we define the **absorption or opacity coefficient** via:

$$dI_\nu = -\kappa_\nu I_\nu ds$$

The inverse of the absorption coefficient is proportional to the mean free path of a photon. Furthermore, as discussed above the probability for absorption of a photon by a **single** atom in state l is:

$$d^3W_{abs} = B_{lu} \varphi(\nu) I_\nu d\nu d\omega dt$$

If we have n_l atoms per volume in level l , we obtain for the number of photons dN_γ^{abs} absorbed in the volume $dV = d\sigma \cdot ds$ along the beam:

$$dN_\gamma^{abs} = n_l \cdot d^3W_{abs} \cdot dV = n_l \cdot B_{lu} \varphi(\nu) I_\nu d\nu d\omega dt d\sigma ds / 4\pi$$

Therefore, the intensity is reduced by:

$$-dI_\nu = \frac{\text{Absorbed Energy}}{d\nu d\omega dt d\sigma} = \frac{h\nu \cdot dN_\gamma^{abs}}{d\nu d\omega dt d\sigma} = n_l \frac{h\nu}{4\pi} B_{lu} \varphi(\nu) I_\nu ds$$

i.e.

$$\kappa_\nu = n_l \frac{h\nu}{4\pi} B_{lu} \varphi(\nu) = n_l \sigma_{lu}$$

where we have used $\sigma_{lu} = h\nu/4\pi \cdot B_{lu} \varphi(\nu)$. This equation relates the absorption coefficient to the microphysical cross section. Including also stimulated emission we have

$$\kappa_\nu = \frac{h\nu}{4\pi} \varphi(\nu) (n_l B_{lu} - n_u B_{ul})$$

or, because of $g_u B_{ul} = g_l B_{lu}$ we can also write:

$$\kappa_\nu = \frac{h\nu}{4\pi} \varphi(\nu) B_{lu} \left(n_l - n_u \frac{g_l}{g_u} \right) = \sigma_{lu} \left(n_l - n_u \frac{g_l}{g_u} \right)$$

If we generalize to all other bound-bound, bound-free and free-free absorption of all elements, and include Thomson scattering, we obtain a formula like:

$$\kappa_\nu = \sum_\alpha \sum_\beta \left\{ \sum_\gamma \left[n_\gamma^{\alpha,\beta} \sigma_{bf,\gamma}^{\alpha,\beta} \left(1 - b_\gamma^{\alpha,\beta} e^{-\frac{h\nu}{kT}} \right) \right. \right. \\ \left. \left. + \sum_{\delta > \gamma} \left(n_\gamma^{\alpha,\beta} - \frac{g_\gamma^{\alpha,\beta}}{g_\delta^{\alpha,\beta}} n_\delta^{\alpha,\beta} \right) \sigma_{\gamma,\delta}^{\alpha,\beta} \right] \right. \\ \left. + n_i^{\alpha,\beta+1} n_e \sigma_{ff}^{\alpha,\beta} \left[1 - e^{-\frac{h\nu}{kT}} \right] \right\}$$

bound - free

lines

free - free

- α = Elements
- β = Degree of ionization
- γ = Levels within degree of ionization
- $\delta > \gamma$ = Levels above
- $n_\gamma^{\alpha,\beta}$ = Occupation numbers
- $\sigma_\gamma^{\alpha,\beta}$ = Absorption cross sections
- $g_\gamma^{\alpha,\beta}$ = Statistical weights of levels
- n_e = Number density (cm^{-3}) of free electrons
- $b_\gamma^{\alpha,\beta}$ = $n_\gamma^{\alpha,\beta}(\text{NLTE}) / n_\gamma^{\alpha,\beta}(\text{LTE}) \approx 1$ (see below)

negative terms = stimulated emission

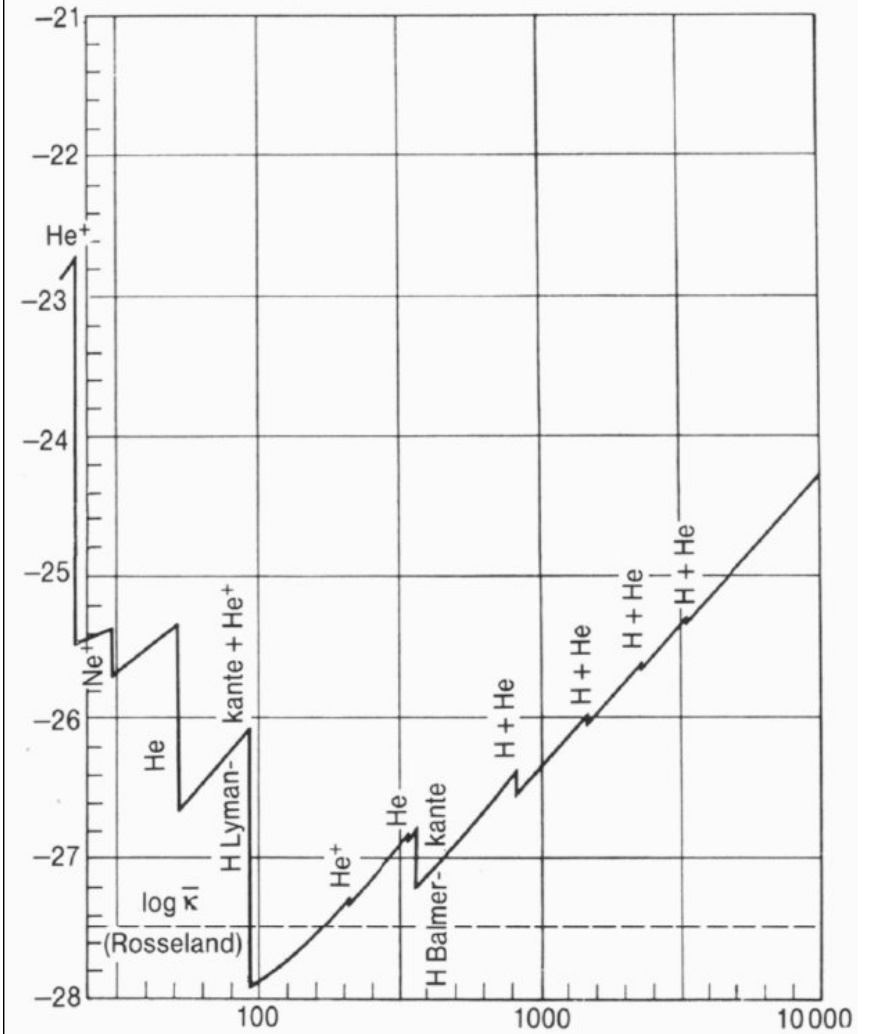
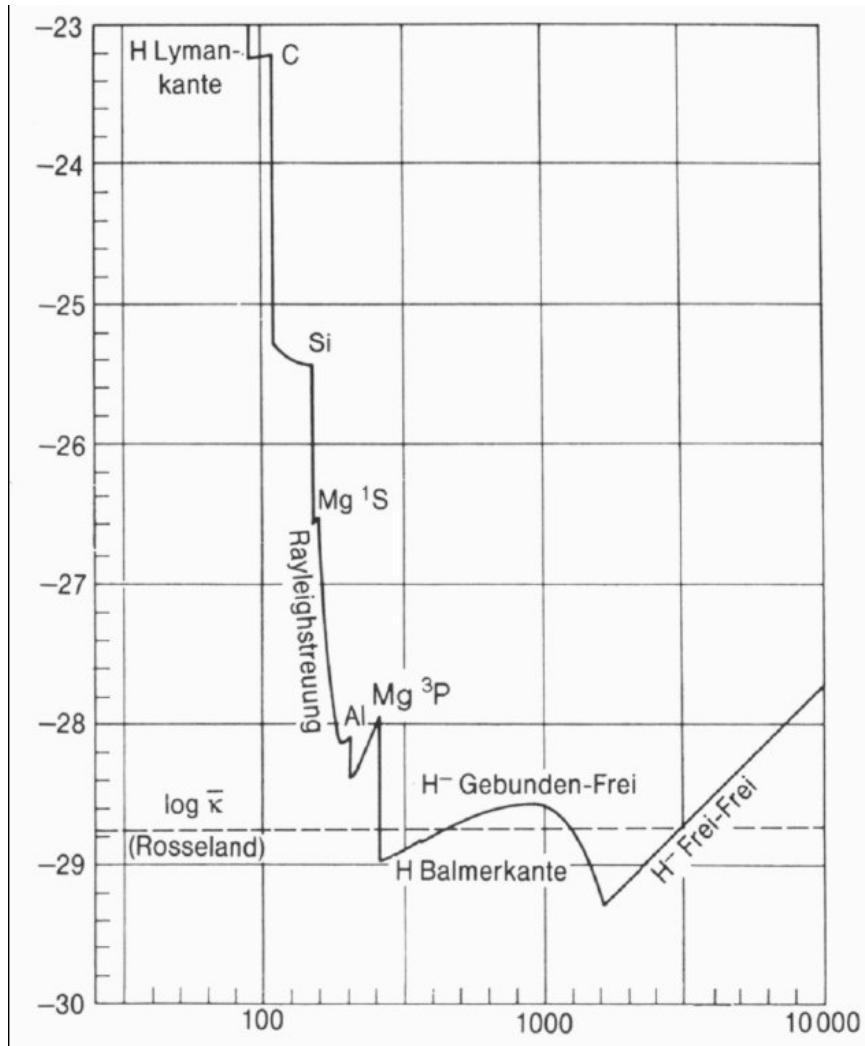
Modern astrophysics takes millions of lines and several 10, 000 bound-free and free-free transitions into account considering almost all elements in all levels of ionization.

- ◆ Important absorption processes in cool stars ($T_{\text{eff}} \leq 8000 \text{ K}$) are:
 - ◆ H^- : bound-free/free-free
 - ◆ HI , AlI , MgI : bound-free
 - ◆ HI and all neutral elements: lines
 - ◆ H_2 , CH , NH , OH , CO , C_2 , CN (diatomic molecules): bound-free/free-free/lines

- ◆ Important absorption processes in hot stars ($T_{\text{eff}} > 8000 \text{ K}$) are:
 - ◆ H , HeI , HeII , abundant metal ions: bound-free, free-free
 - ◆ Free electrons: Thomson scattering
 - ◆ All ions: lines

- ◆ Because the occupation numbers $n_{\nu}^{\alpha, \beta}$ enter κ_{ν} , this is a function of temperature and density: $\kappa_{\nu}(T, \rho)$. For not too large ranges in temperature and density, one can usually separate (see above, relation between κ and cross section):

$$\kappa_{\nu} = \rho^{1 \dots 2} \kappa_{\nu}(T)$$



κ/ρ (in $\text{m}^2/\text{nucleon}$) against λ in nm for the sun ($T = 5000\text{K}$) and τ Sco ($T = 28000\text{K}$).

1.10.2 Emission Coefficient

Spontaneous emission enhances the intensity of radiation or can be the sole source of radiation. Spontaneous emission requires atoms or ions to be in excited states, i.e. they are in a hot plasma or exposed to radiation. We define the emission coefficient via:

$$dI_\nu = \varepsilon_\nu ds$$

Note: dI_ν is independent of I_ν because spontaneous emission does not depend on the incident radiation as stimulated emission (which was therefore included above with the extinction coefficient)

Because of the general Einstein relations between cross sections σ_{abs} and σ_{spont} , we have

$$\varepsilon_\nu = \frac{2h\nu^3}{c^2} \sum_\alpha \sum_\beta \left\{ \sum_\gamma \left[\underbrace{n_\gamma^{\alpha,\beta} b_\gamma^{\alpha,\beta} \sigma_\gamma^{\alpha,\beta} e^{-\frac{h\nu}{kT}}}_{\text{bound - free}} + \underbrace{\sum_{\delta>\gamma} n_\delta^{\alpha,\beta} \frac{g_\gamma^{\alpha,\beta}}{g_\delta} \sigma_{\gamma,\delta}^{\alpha,\beta}}_{\text{lines}} \right] + \underbrace{n_i^{\alpha,\beta+1} n_e \sigma_{ff}^{\alpha,\beta} e^{-\frac{h\nu}{kT}}}_{\text{free - free}} \right\}$$

Therefore we have here as well: $\varepsilon_\nu = \varepsilon_\nu(\rho, T)$.

1.10.3 Radiative Transfer Equation

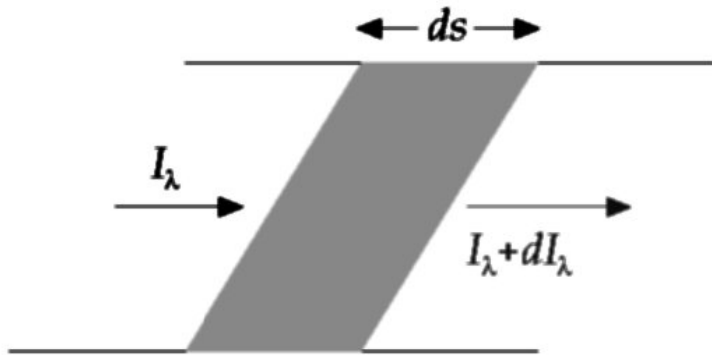
Taking into account absorption, stimulated and spontaneous emission as processes reducing or adding intensity, we finally write the **Equation of Radiative Transfer**:

$$dI_\nu = \underbrace{-\kappa_\nu I_\nu ds}_{\text{absorption}} \quad \underbrace{+\varepsilon_\nu ds}_{\text{emission}}$$

or

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + \varepsilon_\nu$$

This is an inhomogeneous 1st order differential equation which can be solved straightforwardly as long as κ and ε are known as a function of density, temperature and chemical composition and are not dependent on I_ν (as in case of a Laser or the absence of thermodynamical equilibrium).



To solve the equation, we introduce the **optical depth** τ_ν defined via:

$$d\tau_\nu = \kappa_\nu ds \quad \text{or} \quad \tau_\nu = \int_0^s \kappa_\nu ds'$$

We rewrite the equation of radiative transfer as:

$$\frac{dI_\nu}{d\tau_\nu} + I_\nu = \frac{\epsilon_\nu}{\kappa_\nu} = S_\nu$$

where S_ν is called the **source function**. We then multiply by $\exp(\tau_\nu)$ and obtain:

$$\left(\frac{dI_\nu}{d\tau_\nu} + I_\nu \right) e^{\tau_\nu} = \frac{\epsilon_\nu}{\kappa_\nu} e^{\tau_\nu}$$

which is equivalent to:

$$\frac{d}{d\tau_\nu} \left(e^{\tau_\nu} I_\nu \right) = \frac{\epsilon_\nu}{\kappa_\nu} e^{\tau_\nu}$$

of which integration yields:

$$e^{\tau_\nu} \cdot I_\nu - I_\nu^0 = \int_0^{\tau_\nu} \frac{\epsilon_\nu}{\kappa_\nu} e^{\tau'_\nu} d\tau'_\nu$$

or by multiplication with $\exp(-\tau_\nu)$:

$$I_\nu = \underbrace{I_\nu^0 e^{-\tau_\nu}}_{\text{absorption}} + \underbrace{\int_0^{\tau_\nu} \frac{\epsilon_\nu}{\kappa_\nu} e^{-(\tau_\nu - \tau'_\nu)} d\tau'_\nu}_{\text{emission}}$$

This relation can be explained as follows:

I_ν^0 is the radiation falling onto a plane-parallel sheet of plasma with an optical depth of τ_ν . What remains from this incident radiation after the sheet has been crossed is $I_\nu^0 \cdot \exp(-\tau_\nu)$. On the other hand, the sheet emits radiation itself. At each position along the ray, ϵ_ν/κ_ν is contributed. This contribution is however again reduced by the optical depth of the remaining path to the edge of the sheet, i.e. by $\tau_\nu - \tau'_\nu$.

We conclude that:

- Radiation cannot penetrate areas with $\tau_\nu \gg 1$
- If we observe a radiating object, we cannot receive emission from regions with $\tau_\nu \gg 1$ (τ measured inward from the surface of the object).

- For several layers of gas at different temperature (stellar atmosphere!), the observed intensity reflects the temperature at $\tau \approx 1$, because $\tau \approx 1$ corresponds to the mean free path of the photons.

In summary we have:

$$dI_\nu = -\kappa_\nu I_\nu ds + \varepsilon_\nu ds \quad \text{Equation of radiative transfer}$$

$$\tau_\nu = \int_0^s \kappa_\nu ds' \quad \text{Optical depth or optical path}$$

$$S_\nu = \varepsilon_\nu / \kappa_\nu \quad \text{Source Function}$$

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \quad \text{Equation of radiative transfer}$$

$$I_\nu = I_\nu^0 e^{-\tau_\nu} + \int_0^{\tau_\nu} \frac{\varepsilon_\nu}{\kappa_\nu} e^{-(\tau_\nu - \tau'_\nu)} d\tau'_\nu \quad \text{Integral of eq. of radiat. transfer}$$

1.11 Thermodynamical equilibrium and radiative transfer

In perfect radiative equilibrium, we have $I_\nu = B_\nu(T)$ and the radiation field is isotropic and homogeneous, i.e. we also have $dI_\nu/ds = 0$. Under these conditions, the radiative transfer equation reduces to **Kirchhoff's law**:

$$S_\nu(T) = \frac{\varepsilon_\nu}{\kappa_\nu} = B_\nu(T) = \frac{2h\nu^2}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

In principle, only a closed system can be in thermodynamic equilibrium. Therefore, radiative systems, like stars, can never be in a complete **global thermodynamic equilibrium**. Nevertheless, stellar interiors and many other systems are in approximate **local thermodynamical equilibrium (LTE)** if the following two conditions hold:

- Elastic collisions of particles are efficient in establishing a Maxwell velocity distribution with a single local kinetic temperature.
- Inelastic collisions are more frequent than radiative absorption and emission processes and determine the occupation numbers of ionized states and energy levels. This is the case, if the particle density is high enough (see section on LTE below).

Because it does not matter whether the equilibrium occupation numbers are established

via photons or particles, the occupation numbers can be calculated with the Boltzmann and Saha formulae using the local kinetic temperature. Consequently the radiation *emitted* in the small volume in **LTE** also follows a Planck law and we have $S_\nu = B_\nu(T) = \epsilon_\nu/\kappa_\nu$ i.e. again Kirchhoff's law. The total local distribution function of photons is however **not** determined by the local temperature:

$$\frac{dI_\nu}{ds} \neq 0 \quad \Rightarrow \quad I_\nu \neq B_\nu(T)$$

but by the local radiative transfer equation:

$$\frac{dI_\nu}{ds} = -\kappa_\nu \left(I_\nu - B_\nu(T(s)) \right)$$

At any frequency, depending on the difference between incident radiation and locally emitted black body radiation, the intensity either decreases or increases when passing through the small volume in **LTE**.

In the theory of stellar atmospheres, **LTE** is usually assumed (but see exceptions discussed later). As already indicated, this assumption does not mean that we assume complete thermodynamic equilibrium, it only means that we assume that the emission is black-body-radiation of a temperature corresponding to the kinetic temperature of the particles. Even if the incident radiation is concentrated in one line, **LTE** implies that the radiation emitted within a small volume follows $B_\nu(T)$.

1.12 Emission and absorption line spectra

We now discuss under which conditions we will observe emission and absorption line spectra. We consider a box of length s filled with hot gas and located in front of a light source with intensity distribution I_ν^0 . We integrate the equation of radiative transfer through the box:

$$I_\nu = I_\nu^0 e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(T) e^{-(\tau_\nu - \tau'_\nu)} d\tau'_\nu$$

and obtain (as above):

$$I_\nu = I_\nu^0 e^{-\tau_\nu} + S_\nu(T) (1 - e^{-\tau_\nu})$$

or:

$$I_\nu = I_\nu^0 e^{-\tau_\nu} + B_\nu(T) (1 - e^{-\tau_\nu}) \quad \text{in LTE}$$

where τ_ν is the optical depth through the hot gas at frequency ν and S_ν is its source function.

Four limiting cases illustrate what this equation implies:

- $I_\nu^0 \rightarrow 0$, $\tau_\nu \ll 1$, **LTE, i.e. the gas is in LTE but optically thin, and the background illumination is negligible.** Expansion of the exponent in a Taylor series results in:

$$I_\nu = \tau_\nu B_\nu = \kappa_\nu s B_\nu$$

This means that the gas will show strong (weak) emission where κ_ν is large (small). At frequencies corresponding to line transitions between discrete energy levels of the atoms or ions of the gas, κ_ν is large and we will see a characteristic **emission line spectrum**. Astrophysical examples for this situation are: stellar winds, star formation regions, and active galactic nuclei.

- $I_\nu^0 \rightarrow 0$, $\tau_\nu \gg 1$, , i.e. **the gas is in LTE and optically thick, and the background illumination is negligible**. This reduces the equation to:

$$I_\nu = S_\nu \quad \text{or} \quad I_\nu = B_\nu \quad \text{in LTE}$$

The gas in the box emits a featureless **black body spectrum**. Here an astrophysical example is the cosmic microwave background.

- $I_\nu^0 \neq 0$, $\tau_\nu \ll 1$, i.e. **the gas is optically thin, and it is illuminated by a background source**. We have:

$$I_\nu = I_\nu^0(1 - \tau_\nu) + S_\nu \tau_\nu = I_\nu^0 - \kappa_\nu s(I_\nu^0 - S_\nu)$$

$$\Rightarrow I_\nu \downarrow \quad \text{for } S_\nu < I_\nu^0$$

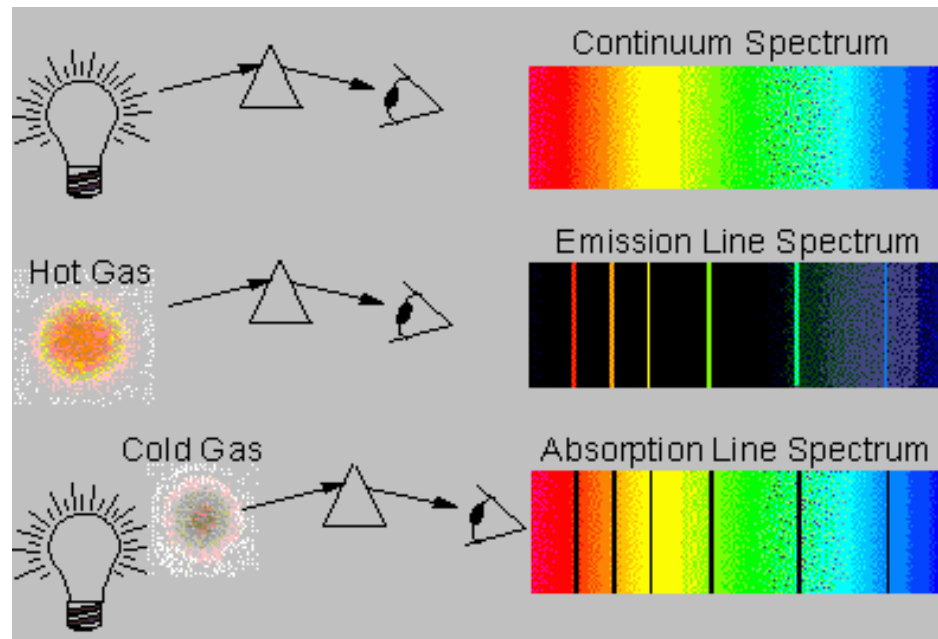
$$I_\nu \uparrow \quad \text{for } S_\nu > I_\nu^0$$

If we have $B_\nu = S_\nu < I_\nu^0$ and κ_ν is large because the frequency corresponds to a line transition in the gas, we evidently obtain an **absorption line** (astrophysical examples: stellar atmospheres, interstellar medium in front of a star, intergalactic gas in front of a quasar).

On the other hand, if $B_\nu = S_\nu > I_\nu^0$ and, again, κ_ν is large because the frequency corresponds to a line transition in the gas, we will see an **emission line on top of the background spectrum**. Stellar coronae are example of this phenomenon.

- $I_\nu^0 \neq 0$, $\tau_\nu \gg 1$, i.e. the gas is optically thick, and it is illuminated by a background source. Here we have again, as in the second case:

$$I_\nu = S_\nu$$



1.13 Validity of Local Thermodynamical Equilibrium

In almost all plasmas (except very thin ones with magnetic fields), the Maxwell distribution is valid. The validity of Saha and Boltzmann formulas on the other hand depend on the ratio of photon energy density u_{photon} to particle energy density $u_{\text{particles}}$

$$u_{\text{photon}} = \frac{4\pi}{c} \int_0^{\infty} B_{\nu} d\nu = \frac{4\pi}{c} \sigma_B T^4$$

$$u_{\text{particle}} = \frac{3}{2} nkT$$

$$\frac{u_{\text{photons}}}{u_{\text{particles}}} = 36.5 \cdot \frac{T^3 / K^3}{n / \text{cm}^{-3}} \begin{cases} \ll 1 & \text{LTE} \\ \sim 1 & \text{LTE questionable} \\ \gg 1 & \text{Non - LTE} \end{cases}$$

$n_{\text{crit}} = 36.5 \cdot T^3$ is the limiting relation for which $u_{\text{photons}} = u_{\text{particles}}$.

If a system is in **Non-LTE** the population of energy levels and ionization states needs to be calculated explicitly on the basis of the radiation field and collisions. The Boltzmann and Saha equations cannot be used anymore. Radiative transfer in this case requires the solution of non-linearly coupled differential equations implying a significant numerical effort.

LTE	LTE questionable	Non - LTE
<ul style="list-style-type: none"> ● Stellar atmospheres ($n = 10^{13} \dots 10^{16} \text{ cm}^{-3}$) for $T_{\text{eff}} \sim 25000 \text{ K}$ ● Stellar cores 	<ul style="list-style-type: none"> ● Atmospheres of super-giants ($n = 10^{10} \dots 10^{13} \text{ cm}^{-3}$) 	<ul style="list-style-type: none"> ● Outer atmospheres (chromosphere, coroneae) ● Stellar winds ● Interstellar medium ● Intergalactic medium