ISSN 1063-7737, Astronomy Letters, 2006, Vol. 32, No. 1, pp. 14–17. © Pleiades Publishing, Inc., 2006.
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Estimating the Stochastization Time in Stellar Systems

A. S. Rastorguev[1*] and V. N. Sementsov[2]**

[1] *Physical Faculty, Moscow State University, Vorob'evy gory, Moscow, 119992 Russia*

[2] *Sternberg Astronomical Institute, Universitetskiĭ pr. 13, Moscow, 119992 Russia*

Received July 12, 2005

Abstract—We show that passage to a statistical description of stellar systems is possible when considering the evolution on time scales longer than $\tau_m \approx \sqrt[5]{\tau_{\rm d}^4 \tau_{\rm c}}$, where $\tau_{\rm d}$ is the mean dynamical (Keplerian) time and $\tau_{\rm c}$ is the two-particle collisional relaxation time.

PACS numbers : 02.50.Ey; 04.40.-b; 05.45.-a; 95.10.Fh; 98.10.+z

DOI: 10.1134/S1063773706010038

Key words: *celestial mechanics, stellar dynamics, relaxation, stochastization, stellar systems.*

INTRODUCTION

The characteristic time scale of "forgetting" the initial conditions plays an important role in studying the dynamics of stellar systems. Indeed, if τ_m is shorter than the evolution time scales considered in a formulated problem, then we can pass to simplifying statistical–mechanical or kinetic descriptions. Otherwise, strictly speaking, we must solve the complete celestial-mechanical problem of the motion of N gravitating bodies and solve it exactly, because there are no *a priori* arguments that the sought-for physical solutions are asymptotically close to the solutions obtained numerically, most commonly in the form of moments of the distribution averaged over the entire $6N$-dimensional phase space. In addition, the procedure for choosing the initial conditions that affect crucially the result becomes much more complicated.

THE CHARACTERISTIC TIME SCALES OF A SYSTEM OF GRAVITATING POINTS

From dimensional considerations (see Dibai and Kaplan 1976), we can obtain two time scales for a system of gravitating points of mass m distributed with a mean space density n and a velocity dispersion v_0^2: the collisional time

$$\tau_{\rm c} = \frac{v_0^3}{(Gm)^2 n} \tag{1}$$

and dynamical time (the crossing time of the stellar system or the Keplerian time)

$$\tau_{\rm d} = (Gmn)^{-1/2}. \tag{2}$$

The results of most studies group around these two estimates. Taking into account only the pair (two-particle) interactions, Chandrasekhar (1948) derived a time of the order of (1), to within a factor that diverges logarithmically in impact parameter. By taking into account only the collisionless collective effects of "violent relaxation," Lynden-Bell (1967) derived a time of the order of (2), to within a factor determined by the Landau damping constant

The ratio of these times can be easily shown to be given by a dimensionless combination of the scale lengths commonly considered in stellar dynamics,

$$\frac{\tau_{\rm d}}{\tau_{\rm c}} = \frac{1}{8}\left(\frac{p_\perp}{d_0}\right)^{3/2}. \tag{3}$$

Here, $p_\perp = 2Gmv_0^{-2}$ is the impact parameter of such a close encounter at which the relative velocity vector of the two stars that had the relative velocity v_0 at a (formally) infinite distance from one another turns through 90° in the frame of reference associated with their center of mass, and $d_0 = 0.5n^{-1/3}$ is the mean distance between the stars in the system. For convenience, we then represent the time scale of interest as

$$\tau_m = \left(\frac{p_\perp}{d_0}\right)^{\alpha} \tau_0, \tag{4}$$

where $\tau_0 = v_0(Gmn^{2/3})^{-1}$ and α is the sought-for parameter. It can be easily shown that $\alpha = -1$ and

*E-mail: rastor@sai.msu.ru
**E-mail: valera@sai.msu.ru

$\alpha = 0.5$ for Eqs. (1) and (2), respectively, and

$$\tau_0^3 = \tau_d^2 \tau_c. \quad (5)$$

The classical approach to estimating τ_m using the *a priori* assumption that the system relaxes to an equilibrium state, which consists in calculating the parameter α in terms of a certain interaction model, involves arbitrariness in choosing the relaxation mechanism and the related parameters and contains a vicious circle (because the relaxation time to equilibrium is calculated by assuming that the equilibrium state is actually reached).

SELF-CONSISTENT ESTIMATION OF THE STOCHASTIZATION TIME

The methods of stochastic dynamics (see Likhtenberg and Liberman 1982), which date back to the classic work by Krylov (2003), serve as a reasonable alternative to this approach. Let us explain the essence of the method without going into mathematical details. A Hamiltonian system with $3N$ degrees of freedom can be represented by a point in $3N$-dimensional (configuration) space. The dynamical evolution of the system can be described by the motion of the representing point along a geodesic curve. Analysis of the properties of the bundle of geodesic lines emerging from a small region of close initial conditions makes it possible to determine whether the properties of stochasticity manifest themselves in the behavior of the system. Indeed, if the geodesic lines diverge rapidly (exponentially), then, provided that the volume accessible for the system is limited (in configuration space), they become greatly entangled and, at a finite observation accuracy, randomly fill the volume almost irrespective of the initial conditions. This phenomenon is called *mixing* and is a property strong enough to prove that the system is ergodic. The absence of mixing is indicative of a relative stability of the motion. Relaxation, in particular, results in the filling of all the phase-volume cells accessible for the system. Therefore, the rate of divergence or the rate of increase of the (coarse) phase volume filled with the representing points corresponding to different initial conditions (its logarithm is the Kolmogorov–Sinai entropy) gives an idea of the *stochastization* time of the system, a constructive analog of the relaxation time (Zaslavsky 1984).

The Hamiltonian of a stellar system is so complex that the equations of geodesic lines cannot be derived in the most general case without using additional assumptions and simplifications. The simplest of them is to postulate a local homogeneity of the stellar system and the Poissonian nature of the appearance of the nearest neighbors of the trial star (which seems to be valid, since the stellar system is fairly sparse). Based on this assumption, Gurzadian and Savvidy (1983, 1984, 1986) and Gurzadian (1998) derived the following formula for the stochastization time:

$$\tau_{GS} = 3.75^{2/3} \left(2\pi\sqrt{2C}\right)^{-1} \tau_0 \approx 0.27 \tau_0 C^{-0.5}, \quad (6)$$

where C is the mean square of the dimensionless force acting on a star in a homogeneous stellar system,

$$C = \int_0^{\beta_{max}} \beta^2 H(\beta) d\beta, \quad (7)$$

calculated from the Holtzmark distribution $H(\beta)$ (see Chandrasekhar 1948). Since the Holzmark distribution diverges at large forces $\beta_{max} \to \infty$ ($r_{min} \to 0$) as $\beta_{max}^{-5/2}$ with $\tau_{GS} \to 0$, it seems natural to limit β_{max} and, accordingly, the minimum distance r_{min}. Gurzadian and Savvidy (1983, 1986) assumed that $r_{min} = p_\perp$ and postulated $C = 1$; they estimated the stochastization time to be $\tau_{GS} \sim \tau_0$.

After correcting the obvious error in Gurzadian and Savvidy (1983) and properly integrating (7), we obtain under the same assumptions

$$C \approx 3e^{-y} y^{-1/3} - \Gamma(2/3) + y^{2/3} \sum_{k=0}^{\infty} \frac{-y^k}{k!(k+2/3)}, \quad (8)$$

where $y = 4\pi(r_{min}/d_0)^3/3$ and $\Gamma(2/3)$ is the Gamma function, or, simplifying this expression for the realistic case of $r_{min} \ll d_0$, we obtain a more accurate value of integral (7),

$$C \approx 2\frac{d_0}{r_{min}}. \quad (9)$$

Following Gurzadian and Savvidy (1983) and assuming that $r_{min} = p_\perp$, we obtain the corrected estimate,

$$\tau'_{GS} \approx 0.125 \left(\frac{p_\perp}{d_0}\right)^{0.5} \tau_0. \quad (10)$$

Under our assumptions, $\tau'_{GS} \approx \tau_d$ should have been considered to be an estimate of the stochasization time. Its low value may appear surprising ($10^{-3}\tau_0$ and $10^{-5}\tau_0$ for a globular cluster and the Galaxy, respectively). Note also that with the above constraint on r_{min}, the energy per unit volume of the system, its square, and other macroscopic quantities averaged over the Holzmark distribution closely match those for a continuous medium. As a matter of fact, under these conditions, the Holzmark distribution yields the *regular* force.

The conflict between the assumptions made and the basic physical principles is the source of the effects mentioned above. Indeed, the mixing being proved suggests the ergodicity of the system as a necessary condition, i.e, the equality between the time-averaged (i.e., measured physical) and ensemble-averaged (calculated) quantities. It is absolutely clear that encounters with impact parameters of the order of $p_\perp$ occur on time scales of the order of the collisional relaxation time, $T_{\rm r} \sim \tau_{\rm c}/100$, which is $\sim 10^{14}$ yr for the Galactic disk. Therefore, the assumption of $r_{\rm min} = p_\perp$ arbitrarily extends the ensemble of systems to include the events (extremely close pair interactions) that *could not* occur in the sought-for time τ_m and should not have showed up when averaging over the time, but, as we see from formulas (7) and (10), make the overwhelming contribution to the estimated force. Hence the error—the implausibly short stochastization time and, as a result, the analysis of hydrodynamic phenomena.

A way out of this situation could be the solution of a partially self-consistent problem, i.e., allowance for only those close interactions (encounters) that can occur on the mixing time scale with a nonnegligible probability. Therefore, we seek for an order-of-magnitude estimate, and we can write the following equation for $r_{\rm min}$ and, hence, for the sought-for mixing time $\tau_m(r_{\rm min})$:

$$\tau_m(r_{\rm min}) v_0 n \pi r_{\rm min}^2 \approx 1 \qquad (11)$$

(clearly, using an approximate equality in this case does not introduce a serious error). The solution of the complete self-consistent problem based on the substitution of the stationary Holzmark distribution with a different, theoretically more justified, distribution of the random force appears impossible.

The solution of Eq. (11) in the asymptotics (9) is

$$r_{\rm min}^5 \approx 6.5 d_0^3 p_\perp^2, \qquad (12)$$

which yields the ultimate formula for τ_m:

$$\tau_m \approx 0.17 \left(\frac{p_\perp}{d_0}\right)^{1/5} \tau_0. \qquad (13)$$

The latter agrees well with the result by Genkin (1972), who estimated $\tau_m \approx \tau_0$ by qualitatively analyzing the "violent" relaxation stage, strictly speaking, based on a similar idea—the attempt to take into account a wider variety of interactions.

In connection with our estimates of the time scale, the results by Petrovskaya (1986) should be mentioned. As is well known, Agekyan (1959) suggested a new method to make allowance for pair interactions. It is based on the treatment of the change in the velocity of a star as an absolutely discontinuous random process. Agekyan gave a formula for the probability of an encounter with a given change in the absolute velocity of a star. Petrovskaya (1986) showed that using the probabilistic approach makes it possible to correct the Holzmark distribution for large dimensionless forces in formula (6) and to ensure the convergence of the integral for the second-order moment, because the corrected distribution has the asymptotics $H_1(\beta) \sim \exp(-1.5\alpha\beta^2)/\beta^3$ at $\beta > 100 Q_{\rm H}$ (the dimensionless force at the mean distance). A simple estimate shows that the relative contribution from encounters with $\beta > 100 Q_{\rm H}$ for realistic $r_{\rm min}/d_0 \sim 10^{-3}$–$10^{-4}$ does not exceed 2×10^{-3}, which does not change our conclusions at all.

CONCLUSIONS

Representing Eq. (13) in terms of the commonly used time scales $\tau_{\rm c}$ (1) and $\tau_{\rm d}$ (2) may prove to be more convenient. Using Eqs. (3) and (5), we obtain

$$\tau_m \approx 0.224 \sqrt[5]{\tau_{\rm d}^4 \tau_{\rm c}}. \qquad (14)$$

In a spatially homogeneous system (which is an additional strong assumption), the ratio $\tau_{\rm c} : \tau_{\rm d}$ depends on the total number N of stars in the system, which allows formula (14) to be simplified:

$$\tau_m \approx \tau_{\rm d} \sqrt[5]{N}. \qquad (15)$$

Comparison of the τ_m values in a globular cluster ($\tau_m \sim 10^6$–10^7 yr) and the Galactic halo ($\tau_m \sim 10^{10}$–10^{11} yr) with the cosmological time (10^{10} yr) shows that applying the statistical–mechanical methods to stellar systems is quite justifiable from the viewpoint that an equilibrium (in the sense of filling the phase volume) is established in them in a time shorter than the age.

Note, however, that the stochastization of motions in stellar systems is not relaxation in the sense this term is applied to ordinary gases or plasma. On time scales longer than τ_m, a stellar system is described by a number of parameters that is much smaller than the number of mechanical parameters, $6N$, but larger than that in the statistical mechanics of an ideal gas, especially when analyzing systems with a common angular momentum.

ACKNOWLEDGMENTS

This work was supported in part by the Russian Foundation for Basic Research (project no. 05-02-16526) and the "Program for Support of Leading Scientific Schools" (project no. NSh-389.2.2003).

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Translated by A. Dambis