СТАЦИОНАРНЫЕ ДВИЖЕНИЯ ТРЕХ ВЯЗКОУПРУГИХ ПЛАНЕТ

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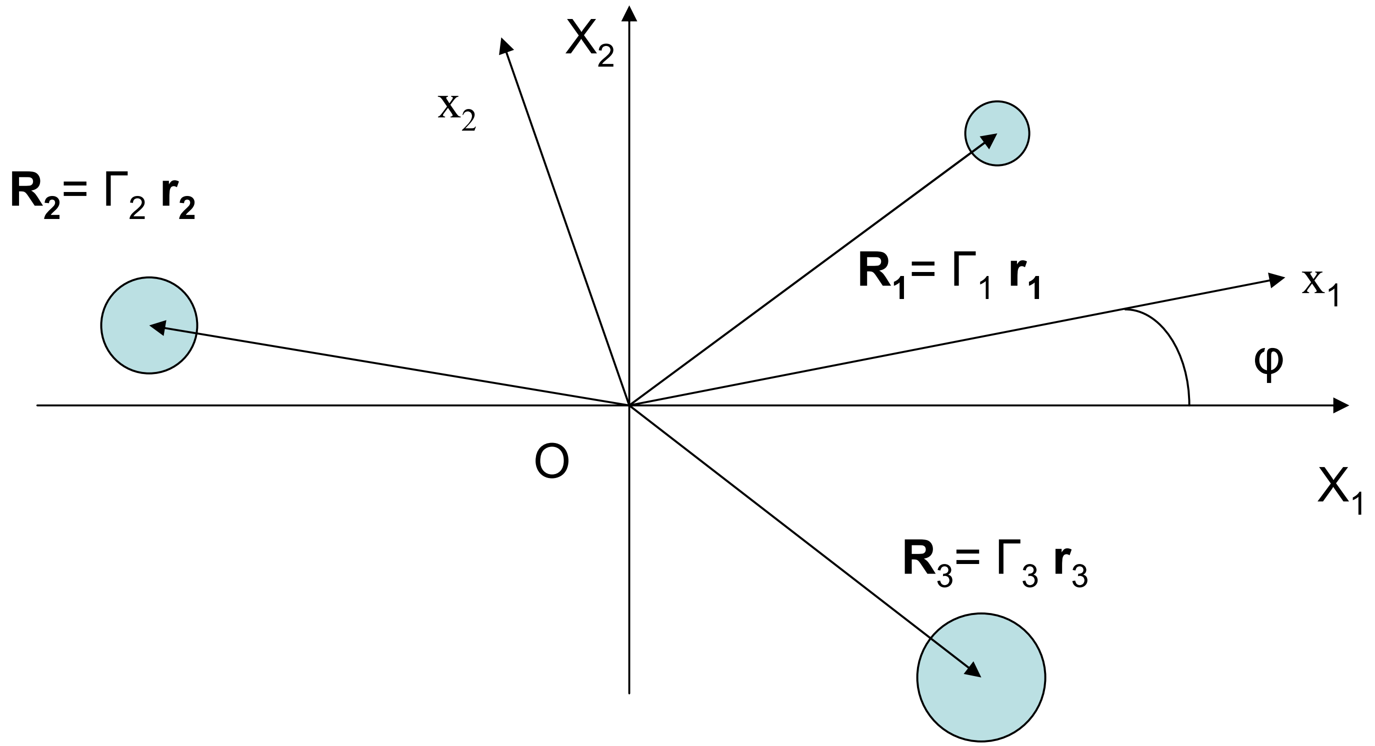
Системы отсчета и уравнения движения



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{R}}_i}\right)-\frac{\partial L}{\partial \mathbf{R}_i}=0, \qquad \dot{\mathbf{L}}_i=\frac{\partial L}{\partial \boldsymbol{\alpha}_i}, \qquad i=1,2,3 \tag{1}$$

$$L=\frac{1}{2}\sum_{i=1}^{3}[m_i\dot{\mathbf{R}}_i^2+A_i\boldsymbol{\omega}_i^2]-\sum_{i<j}^{3}\frac{\gamma m_i m_j}{R_{ij}}-\Pi_p, \quad \boldsymbol{\alpha}_i=(\alpha_{1i},\alpha_{2i},\alpha_{3i}), \ \mathbf{L}_i=\frac{\partial L}{\partial \boldsymbol{\omega}_i} \tag{2}$$

$$\Pi_p=\sum_{i=1,j\neq i}^{3}h_i m_j\left[\frac{3(\Gamma_i\boldsymbol{\omega}_i,\mathbf{R}_{ij})^2}{R_{ij}^5}-\frac{\boldsymbol{\omega}_i^2}{R_{ij}^3}\right]+\frac{3\gamma}{2}\sum_{i=1,j\neq i,k\neq i}^{3}\frac{h_i m_j m_k}{R_{ij}^3R_{ik}^3}\left[1-\frac{3(\mathbf{R}_{ij},\mathbf{R}_{ik})^2}{R_{ij}^2R_{ik}^2}\right]$$

$$R_{ij}=\left|\mathbf{R}_i-\mathbf{R}_j\right| \qquad h_i=\frac{4\pi\gamma\rho_i^2a_i^7(1+\nu_i)(9\nu_i+13)}{105E_i(5\nu_i+7)}=\frac{\gamma m_i\rho_i a_i^4(1+\nu_i)(9\nu_i+13)}{35E_i(5\nu_i+7)}$$

$$\varepsilon_{ij} = \rho_i \gamma m_j R_{ij}^{-3} a_i^2 E_i^{-1} \qquad \varepsilon_{i0} = \rho_i \boldsymbol{\omega}_i^2(0)\, a_i^2 E_i^{-1}$$

$$\mathbf{G} = \sum_{i=1}^{N} (\mathbf{L}_i + [\mathbf{R}_i \times m_i \dot{\mathbf{R}}_i]) = G\,\mathbf{e}_3, \quad G = const$$

Сделаем замену переменных

$$\mathbf{R}_i = \Gamma_3(\varphi)\mathbf{r}_i, \ \dot{\mathbf{R}}_i = \Gamma_3(\varphi)\{\dot{\varphi}[\mathbf{e}_3 \times \mathbf{r}_i] + \dot{\mathbf{r}}_i\}, \ \Gamma_i \boldsymbol{\omega}_i = \dot{\varphi}\mathbf{e}_3 + \Gamma_3(\varphi)\mathrm{B}_i \boldsymbol{\nu}_i$$

$$L = \frac{1}{2}\sum_{i=1}^{3}[m_i(\dot{\varphi}[\mathbf{e}_3 \times \mathbf{r}_i] + \dot{\mathbf{r}}_i)^2 + A_i(\dot{\varphi}\mathbf{e}_3 + \mathrm{B}_i \boldsymbol{\nu}_i)^2] - \sum_{i<j}^{3} \frac{\gamma m_i m_j}{r_{ij}} - \Pi_p$$

$$\Pi_p = \sum_{i=1, j\neq i}^{3} h_i m_j \left[\frac{3((\dot{\varphi}\mathbf{e}_3 + \mathrm{B}_i \boldsymbol{\nu}_i), \mathbf{r}_{ij})^2}{r_{ij}^5} - \frac{(\dot{\varphi}\mathbf{e}_3 + \mathrm{B}_i \boldsymbol{\nu}_i)^2}{r_{ij}^3} \right] + \frac{3\gamma}{2} \sum_{i=1, j\neq i, k\neq i}^{3} \frac{h_i m_j m_k}{r_{ij}^3 r_{ik}^3} \left[1 - \frac{3(\mathbf{r}_{ij}, r_{ik})^2}{r_{ij}^2 r_{ik}^2} \right]$$

Функция Рауса и уравнения Рауса

$$L = F_2\,\dot{\varphi}^2/2 + F_1\,\dot{\varphi} + F_0 \qquad\qquad \frac{\partial L}{\partial\dot{\varphi}} = F_2\,\dot{\varphi} + F_1 = G$$

$$\Re = \frac{\partial L}{\partial\dot{\varphi}}\cdot\dot{\varphi} - L = \frac{1}{2}F_2\,\dot{\varphi}^2 - F_0 = \frac{(G - F_1)^2}{2F_2} - F_0 = \frac{F_1^2}{2F_2} - \frac{G\,F_1}{F_2} + \frac{G^2}{2F_2} - F_0$$

$$F_2 = \frac{\partial^2 L}{\partial\dot{\varphi}^2} = \sum_{i=1}^{3}(m_i[\mathbf{e}_3\times\mathbf{r}_i]^2 + A_i) + \sum_{i=1,\,j\neq i}^{3} 2h_i\,m_j\left[\frac{3(\mathbf{e}_3,\mathbf{r}_{ij})^2}{r_{ij}^5} - \frac{1}{r_{ij}^3}\right]$$

$$F_0 = \frac{1}{2}\sum_{i=1}^{3}[m_i\,\dot{\mathbf{r}}_i^{\,2} + A_i\mathbf{v}_i^{\,2}] + \sum_{i<j}^{3}\frac{\gamma\,m_i\,m_j}{r_{ij}} - \sum_{i=1,\,j\neq i}^{3} h_i\,m_j\left[\frac{3(\mathrm{B}_i\mathbf{v}_i,\mathbf{r}_{ij})^2}{r_{ij}^5} - \frac{\mathbf{v}_i^{\,2}}{r_{ij}^3}\right] +$$

$$+\frac{3\gamma}{2}\sum_{i=1,\,j\neq i,\,k\neq i}^{3}\frac{h_i\,m_j\,m_k}{r_{ij}^3 r_{ik}^3}\left[1 - \frac{3(\mathbf{r}_{ij},\mathbf{r}_{ik})^2}{r_{ij}^2 r_{ik}^2}\right]$$

Измененная потенциальная энергия. Эйлеровы и лагранжевы стационарные конфигурации

$$V(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\frac{G^2}{2F_2}-\sum_{i<j}^{3}\frac{\gamma\, m_i\, m_j}{r_{ij}}+\frac{3\gamma}{2}\sum_{i=1,\, j\neq i,\, k\neq i}^{3}\frac{h_i\, m_j\, m_k}{r_{ij}^3 r_{ik}^3}\left[1-\frac{3(\mathbf{r}_{ij},\mathbf{r}_{ik})^2}{r_{ij}^2 r_{ik}^2}\right]$$

$$h^{\alpha}:(x_{1i},x_{2i},x_{3i})\mapsto(x_{1i},x_{2i},\alpha\, x_{3i})\,,\ \alpha\in]-\alpha_0,\,\alpha_0\,[,\ \ i=1,2,3$$

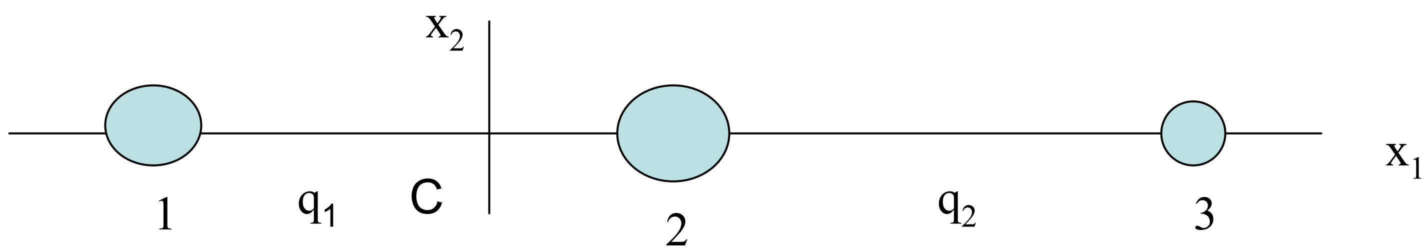
$$r_{ij}=[(x_{1i}-x_{1j})^2+(x_{2i}-x_{2j})^2+\alpha^2(x_{3i}-x_{3j})^2]^{1/2}$$

$$(\mathbf{r}_{ij},\mathbf{r}_{ik})=(x_{1i}-x_{1j})(x_{1k}-x_{1k})+(x_{2i}-x_{2j})(x_{2k}-x_{2k})+\alpha^2(x_{3i}-x_{3j})(x_{3k}-x_{3k})$$

$$V(h^{\alpha}\mathbf{r}_1,\,h^{\alpha}\mathbf{r}_2,h^{\alpha}\mathbf{r}_3)=\Psi(\alpha^2)\ \Rightarrow\frac{dV(h^{\alpha}\mathbf{r}_1,\,h^{\alpha}\mathbf{r}_2,h^{\alpha}\mathbf{r}_3)}{d\,\alpha}=\frac{d\,\Psi(\alpha^2)}{d\,(\alpha^2)}2\alpha$$

Эйлеровы стационарные конфигурации

$$\sum_{i=1}^{3} m_i [\mathbf{e}_3 \times \mathbf{r}_i]^2 = M^{-1} \sum_{i<j}^{3} m_i m_j r_{ij}^2, \quad M = m_1 + m_2 + m_3$$



$$V(q_1, q_2) = \frac{G^2}{2I} - \frac{G^2}{I^2} \sum_{i<j}^{3} \frac{h_i m_j + h_j m_i}{r_{ij}^3} - \sum_{i<j}^{3} \frac{\gamma m_i m_j}{r_{ij}} - 3\gamma \sum_{(i,j,k)}^{3} h_i \left(\frac{m_j}{r_{ij}^3} + \frac{m_k}{r_{ik}} \right)^2$$

$$I = M^{-1} \sum_{i<j}^{3} m_i m_j r_{ij}^2 + A_1 + A_2 + A_3, \qquad r_{ij} = r_{ji}$$

$$r_{12} = q_1, \; r_{23} = q_2, \; r_{13} = q_1 + q_2$$

$$\frac{\partial V\,(q_1, q_2)}{\partial\, q_1} = 0, \qquad \frac{\partial V\,(q_1, q_2)}{\partial\, q_2} = 0$$

$$\delta V(q_1,q_2) = -\frac{G^2}{2I^2}\delta I + \frac{2G^2}{I^3}\delta I \sum_{i<j}^{3}\frac{h_i m_j + h_j m_i}{r_{ij}^3} + \frac{3G^2}{I^2}\sum_{i<j}^{3}\frac{h_i m_j + h_j m_i}{r_{ij}^4}\delta r_{ij} +$$

$$+\sum_{i<j}^{3}\frac{\gamma m_i m_j}{r_{ij}^2}\delta r_{ij} + 18\gamma \sum_{(i,j,k)}^{3} h_i\left(\frac{m_j}{r_{ij}^3}+\frac{m_k}{r_{ik}^3}\right)\left(\frac{m_j}{r_{ij}^4}\delta r_{ij} + \frac{m_k}{r_{ik}^4}\delta r_{ik}\right)$$

$$\delta I = 2M^{-1}\sum_{i<j}^{3} m_i m_j r_{ij}\delta r_{ij}, \qquad \delta r_{12} = \delta q_1,\ \delta r_{13} = \delta q_1 + \delta q_2,\quad \delta r_{23} = \delta q_2$$

$$\delta^2 V(q_1,q_2) = -\frac{G^2}{2I^2}\delta^2 I + \frac{2G^2}{I^3}\delta^2 I \sum_{i<j}^{3}\frac{h_i m_j + h_j m_i}{r_{ij}^3} - \frac{12G^2}{I^2}\sum_{i<j}^{3}\frac{h_i m_j + h_j m_i}{r_{ij}^5}(\delta r_{ij})^2 -$$

$$-2\sum_{i<j}^{3}\frac{\gamma m_i m_j}{r_{ij}^3}(\delta r_{ij})^2 - 72\gamma \sum_{(i,j,k)}^{3} h_i\left(\frac{m_j}{r_{ij}^3}+\frac{m_k}{r_{ik}^3}\right)\left(\frac{m_j}{r_{ij}^5}(\delta r_{ij})^2 + \frac{m_k}{r_{ik}^5}(\delta r_{ik})^2\right) -$$

$$-54\gamma \sum_{(i,j,k)}^{3} h_i\left(\frac{m_j}{r_{ij}^4}\delta r_{ij} + \frac{m_k}{r_{ik}^4}\delta r_{ik}\right)^2, \ \delta^2 I = 2M^{-1}\sum_{i<j}^{3} m_i m_j (\delta r_{ij})^2, \ \delta^2 r_{12} = \delta^2 r_{13} = \delta^2 r_{23} = 0$$

$$\delta I = 2M^{-1}[m_1 m_2 q_1 \delta q_1 + m_2 m_3 q_2 \delta q_2 + m_1 m_3 (q_1 + q_2)(\delta q_1 + \delta q_2)] = 0 \qquad I > 4\sum_{i<j}^{3}\frac{h_i m_j + h_j m_i}{r_{ij}^3}$$

Треугольные стационарные конфигурации

$$V(\mathbf{q}) = V_0(\mathbf{q}) - H(\mathbf{q}), \qquad V_0(\mathbf{q}) = \frac{G^2}{2I} - \sum_{(i,j,k)} \frac{\gamma m_i m_j}{q_k}$$

$$H(\mathbf{q}) = \frac{G^2}{I^2} \sum_{(i,j,k)} \frac{h_i m_j}{q_k^3} + 3\gamma \sum_{(i\,j\,k)} h_i \left\{ \left(\frac{m_j}{q_k^3} + \frac{m_k}{q_j^3} \right)^2 + \frac{3m_j m_k}{q_j^3 q_k^3} \left[\frac{(q_j^2 + q_k^2 - q_i^2)^2}{4q_j^2 q_k^2} - 1 \right] \right\}$$

$$I = M^{-1} \sum_{(i,j,k)} m_i m_j q_k^2 + A_1 + A_2 + A_3$$

.

$$\mathbf{q} = (q_1, q_2, q_3), \; q_1 = r_{23}, \; q_2 = r_{31}, \; q_3 = r_{12}$$

$$\frac{\partial V(\mathbf{q})}{\partial q_k} = 0 \;\Rightarrow\; \frac{\partial V_0(\mathbf{q})}{\partial q_k} = \frac{\partial H(\mathbf{q})}{\partial q_k}, \qquad k = 1,2,3$$

$$h_i << 0, \; i = 1,2,3$$

$$\delta^2 V_0 = -\frac{G^2}{M I^2} \sum_{(i\,j\,k)} m_i m_j (\delta q_k)^2 - \sum_{(i\,j\,k)} \frac{\gamma m_i m_j}{q_k^3} (\delta q_k)^2 \le 0$$

$$\delta I(\mathbf{q}) = 2M^{-1} \sum_{(i\,j\,k)} m_i m_j q_k \delta q_k = 0$$

Возмущения сторон равностороннего треугольника

$$\frac{\partial V_0(\mathbf{q})}{\partial q_k} = -\frac{G^2}{M\, I^2(\mathbf{q})} m_i\, m_j\, q_k + \frac{\gamma\, m_i\, m_j}{q_k^2} = 0 \qquad (i\, j\, k)$$

$$q_1 = q_2 = q_3 = R \qquad G^2 R^3 = \gamma\, M\, [M^{-1}(m_1 m_2 + m_2 m_3 + m_3 m_1) R^2 + A]^2$$

$$G_*^4 = \frac{256\ \gamma^2\ A\,(m_1 m_2 + m_2 m_3 + m_3 m_1)}{27\ M}$$

$$q_i = R + u_i \qquad \sum_{s=1}^{3} \frac{\partial^2 V_0(\mathbf{R})}{\partial q_k \partial q_s} u_s = \frac{\partial H(\mathbf{R})}{\partial q_k}, \ \ k = 1,2,3; \quad \mathbf{R} = (R, R, R)$$

$$\frac{\partial^2 V_0(\mathbf{R})}{\partial q_k \partial q_s} = \frac{m_i\, m_j G^2}{M\ \ I^2(\mathbf{R})}\left[\frac{4 m_p m_n\ R^2}{M\ \ I(\mathbf{R})} - 3\delta_{ks}\right], \quad (i\ j\ k), \quad (p\ n\ s)$$

$$I(\mathbf{R}) = M^{-1}(m_1 m_2 + m_2 m_3 + m_3 m_1) R^2 + A$$

$$\frac{\partial H(\mathbf{R})}{\partial q_k} = -\frac{\gamma}{R^7}[3M(h_i m_j + h_j m_i) + 4R^2 I^{-1} m_i m_j (h_1 m_2 + h_2 m_3 + h_3 m_1) +$$

$$+ h_i(18 m_j^2 - \frac{27}{4} m_j m_k) + h_j(18 m_i^2 - \frac{27}{4} m_i m_k) + 9 h_k m_i m_j], \quad (i\ j\ k)$$

Приближенное решение для большего корня невозмущенной задачи

$$I(\mathbf{R}) \approx M^{-1}(m_1 m_2 + m_2 m_3 + m_3 m_1) R^2 = m R^2 \qquad m_1 = m_2 = m_3 = m$$

$$\frac{\partial^2 V_0(\mathbf{R})}{\partial q_k \partial q_s} = \frac{\gamma m^2}{R^3}\left[\frac{4}{3} - 3\delta_{ks}\right], \quad \frac{\partial H(\mathbf{R})}{\partial q_k} = -\frac{\gamma m^2}{4R^7}(97 h_i + 97 h_j + 52 h_k), (i\ j\ k)$$

$$\begin{pmatrix} 5 & -4 & -4 \\ -4 & 5 & -4 \\ -4 & -4 & 5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{3}{4R^4}\begin{pmatrix} 194h_1 + 149h_2 + 149h_3 \\ 149h_1 + 194h_2 + 149h_3 \\ 149h_1 + 149h_2 + 194h_3 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = -\frac{1}{4R^4}\begin{pmatrix} 154h_1 + 169h_2 + 169h_3 \\ 169h_1 + 154h_2 + 169h_3 \\ 169h_1 + 169h_2 + 154h_3 \end{pmatrix}$$

Диссипативный функционал и диссипативная функция

$$W=\sum_{i=1}^{N}\chi_i\,\mathrm{E}_i[\sum_{j\neq i}^{N}\varepsilon_i\,\dot{\mathbf{v}}_{ij}(\mathbf{r}_i,t)]=\frac{1}{2}\sum_{i=1}^{N}\chi_i\varepsilon_i\,(\sum_{j\neq i}^{N}\nabla_{\dot{\mathbf{v}}_{ij}}\mathrm{E}_i[\dot{\mathbf{v}}_{ij}(\mathbf{r}_i,t)],\sum_{s\neq i}^{N}\dot{\mathbf{v}}_{is}(\mathbf{r}_i,t))\;=$$

$$=\frac{\gamma^2}{2}\sum_{i=1}^{N}\chi_i\varepsilon_i\,\rho_i^2\sum_{j,s\neq i}^{N}\{\int_{V_i}[a_{1i}\,(D_{ij}\,\mathbf{r}_i,\mathbf{r}_i)(D_{is}\,\mathbf{r}_i,\mathbf{r}_i)+(a_{2i}\,\mathbf{r}_i^2+a_{3i})(D_{ij}\,\mathbf{r}_i,D_{is}\mathbf{r}_i)]d\,x^{(i)}\}$$

$$D_{ij}=m_j\frac{d}{dt}(R_{ij}^{-3}\,B_{ij})\,,\quad B_{ij}=(3e_{k,ij}\,e_{l,ij}-\delta_{kl})$$

$$\Gamma_i^{-1}\,\mathbf{R}_{ij}^{\circ}=\mathbf{e}_{ij}=(e_{1,ij},\,e_{2,ij},\,e_{3,ij}).$$